DIRAC EQUATION WITH EUCLIDEAN-MINKOWSKIAN "GRAVITY PHASE"

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Abstract. Starting with the conventional Dirac equation and a $\frac{1}{r}$-type potential, a one-parameter real phase α is introduced that transitions between Euclidean ($\alpha = 0, \pm\pi, \pm 2\pi, \ldots$) and Minkowskian ($\alpha = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$) geometry. Spin-$\frac{1}{2}$ Coulomb scattering (Rutherford scattering) in Born approximation is executed. All calculations are done "from scratch" as they could have been done 80 years ago, i.e., without any elegance from field theory that has since become customary. Results for backscattering and cross section at high energies are compared with results from General Relativity calculations. Effects on intergalactic gas distribution and momentum transfer from scattering high-energy leptons are sketched. This work follows discussions at the 4th Mile High Conference on Nonassociative Mathematics (2017), University of Denver, CO.

Contents

1. Introduce phase α to the Dirac equation

Start with the conventional Dirac equation and a $\frac{1}{r}$-type potential. Introduce a one-parameter real phase α that transitions between Euclidean ($\alpha = 0, \pm\pi, \pm 2\pi, \ldots$) and Minkowskian ($\alpha = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$) geometry.

1.1. **Define phase α.** Let $\alpha \in \mathbb{R}$ be the phase angle that transitions between Euclidean ($\alpha = 0, \pm\pi, \pm 2\pi, \ldots$) and Minkowskian ($\alpha = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$) geometry.

Define two copies of the complex numbers, $\mathbb{C}$ and $\mathbb{C}_0$, to basis elements $\{1, i\}$ and $\{1, i_0\}$, respectively. Dedicate $\mathbb{C}_0$ to modeling the phase from α, and $\mathbb{C}$ for modeling conventional Dirac equation and operators. Elements from $\mathbb{C}$ and $\mathbb{C}_0$ commute, associate, and distribute just like real coefficients.

Given a variable x, complex conjugation in $\mathbb{C}$ is written as $\overline{x}$ and complex conjugation in $\mathbb{C}_0$ as $\underline{x}$.

Define phase ϕ and conjugate $\underline{\phi}$ as:

$$\phi := e^{i_0\alpha}, \qquad \underline{\phi} := e^{-i_0\alpha}, \qquad \phi\underline{\phi} = \underline{\phi}\phi = 1. \tag{1.1}$$

1.2. **Introduce α to Dirac equation.** Using 4×4 matrices over the complexes, γ_μ, with $\mu = 0 \ldots 3$, linear derivatives $\partial_\mu := \frac{\partial_\mu}{\partial x_\mu}$, a property $\widetilde{m}$ that is invariant under ∂_μ (with $\widetilde{m}|_{\text{QED}} \equiv m \in \mathbb{R}$ mass in the classical case), and functions $\psi : \mathbb{R}^4 \to \mathbb{C}^4$, the Dirac equation is written as the eigenvalue relation:

$$\sum_{\mu=0}^{3} i\gamma_\mu\partial_\mu\psi = \widetilde{m}\psi. \tag{1.2}$$

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In contrast to notation convention in physics today, all indices are now written as lower indices, and summation is written explicitly (i.e. without implicit Minkowski tensor). Spelling out summations and metric explicitly avoids confusion going forward, when Minkowski metric is considered an edge case in a generalized geometry.

Generalized Dirac matrices, using the same symbol γ_μ, are now defined as a function of α, to model the conventional Dirac equation in the $\alpha = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$ cases and a counterpart on Euclidean metric in the $\alpha = 0, \pm\pi, \pm 2\pi, \ldots$ case (per [1, 2]):

$$(1.3)\quad \gamma_0 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \gamma_1 := \begin{pmatrix} 0 & 0 & 0 & \phi^2 \\ 0 & 0 & \phi^2 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \gamma_2 := \begin{pmatrix} 0 & 0 & 0 & -i\phi^2 \\ 0 & 0 & i\phi^2 & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \gamma_3 := \begin{pmatrix} 0 & 0 & \phi^2 & 0 \\ 0 & 0 & 0 & -\phi^2 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

With $\gamma_\mu \in (\mathbb{C} \times \mathbb{C}_0)^4$ the wave functions ψ are now generally $\psi : \mathbb{R}^4 \to (\mathbb{C} \times \mathbb{C}_0)^4$.

Using Pauli spinors σ_j with $j = 1, 2, 3$

$$(1.4)\quad \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and identifying the unit matrix with

$$(1.5)\quad I_2 \equiv \sigma_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

the generalized γ_μ can be written as:

$$(1.6)\quad \gamma_0 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix} = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \gamma_j = \begin{pmatrix} 0 & \phi^2\sigma_j \\ \sigma_j & 0 \end{pmatrix}.$$

It is left open for now whether this requires the parameter $\widetilde{m}$ to become complex in $\mathbb{C}_0$ or not.

1.3. **Properties of the generalized Dirac-γ.** The Euclidean and Minkowskian edge cases from the referenced papers ([1, 2]) are satisfied by inspection, for $\phi^2 = 1$ and $\phi^2 = -1$, respectively.

Writing I_4 for the identity 4×4 matrix, the generalized Dirac matrices have the property:

$$(1.7)\quad \frac{1}{2}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu) = I_4 * \begin{cases} 0 & \text{for } \mu \neq \nu, \\ 1 & \text{for } \mu = \nu = 0, \\ \phi^2 & \text{otherwise } (\mu = \nu \in \{1, 2, 3\}). \end{cases}$$

This property reflects the choice of metric.

1.4. **Energy, mass, momentum, relativity.** Understanding the derivatives $i\partial_\mu$ as quantum mechanical operators for energy E and momentum $\vec{p} := (p_1, p_2, p_3)$, $|\vec{p}|^2 := p_1^2 + p_2^2 + p_3^2$, multiplying the Dirac equation (1.2) with its conjugate in $\mathbb{C}$ recovers the Minkowskian $|E|^2 - |\vec{p}|^2 = m^2_{\text{Mink}}$ and Euclidean $|E|^2 + |\vec{p}|^2 = m^2_{\text{Eucl}}$ edge cases for $\phi^2 = 1$ and $\phi^2 = -1$, respectively.

In general, the sum over four momentum p,

$$(1.8)\quad p := (p_\mu) = (E, \vec{p}) = (E, p_1, p_2, p_3), \qquad \text{with } E, \vec{p} \in \mathbb{R}.$$

becomes:

$$(1.9)\quad m_0^2 := \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} \gamma_\mu\gamma_\nu p_\mu p_\nu = E^2 + \phi^2 |\vec{p}|^2.$$

This makes m_0^2 complex in $\mathbb{C}_0$ in the general case. In order to keep the classical mass parameter $m_\mathbb{R} \in \mathbb{R}$ real, it has to be redefined as compared to the conventional case. The simplest way is to take the absolute of m_0^2, which makes $m_\mathbb{R}$ a fourth-order expression in E and $\vec{p}$:

$$(1.10)\quad \phi^2 = e^{2i_0\alpha}, \text{ therefore}$$

$$(1.11)\quad m_\mathbb{R}^4 : = |m_0^2|^2 = E^4 + 2E^2 |\vec{p}|^2 \cos(2\alpha) + |\vec{p}|^4,$$

$$(1.12)\quad m_\mathbb{R} = \sqrt{|m_0^2|} = \sqrt[4]{E^4 + 2E^2 |\vec{p}|^2 \cos(2\alpha) + |\vec{p}|^4}.$$

There are both physical and mathematical implications to this.

1.4.1. *Physical implications.* From the physical side, it raises the question on what makes physical lab frames equivalent. Next to energy and momentum of test bodies in unaccelerated frames of reference, the parameter α now factors into the equivalence condition as well. Equation 1.12 therefore becomes the new definition of relativity, as proposed in [3, 4]. This requires clarification of the meaning of α, which is subject of this current research.

The relation between speed $\vec{v}$, energy E, and momentum $\vec{p}$ remains unchanged for any α by definition:

$$(1.13)\quad \vec{v} := \frac{\vec{p}}{E}.$$

Conserved properties between equivalent frames of reference are generalized per equation (1.9) and therefore complex in $\mathbb{C}_0$ in the general case. For example, invariant length elements ℓ_0, volumes V_0, or time intervals $d\tau$:

$$\ell_0 := \ell\frac{E}{m_0} = \frac{\ell}{\sqrt{1+\phi^2\left|\vec{v}\right|^2}}, \qquad V_0 := V\frac{E}{m_0} = \frac{V}{\sqrt{1+\phi^2\left|\vec{v}\right|^2}}, \qquad d\tau := dt\frac{m_0}{E} = dt\sqrt{1+\phi^2\left|\vec{v}\right|^2}. \tag{1.14}$$

1.4.2. *Mathematical implications.* From the mathematical side, fourth-order expressions are a departure from the pure (Dirac-)spinor expression of the equation of motion of a spin-$\frac{1}{2}$ particle. It requires clarification on what mathematical construct we're looking at exactly - or alternatively, find a more natural mathematical representation of the phase α. This may ultimately lead to a more natural mathematical description of relativity, which might not be apparent here due to the focus on a special case.

For now it is still left open whether the placeholder $\widetilde{m}$ in the generalized Dirac equation is the classical, real-valued mass parameter $m_{\mathbb{R}}$, or whether it has to become m_0 which is complex-valued in $\mathbb{C}_0$.

1.5. **Green's function in energy-momentum space.** In momentum space, Green's function $G(p)$ solves

$$\left(\sum_{\mu=0}^{3}\gamma_\mu p_\mu - \widetilde{m}\right) G(p) = 1 \tag{1.15}$$

and is:

$$G(p) = \frac{\sum_{\mu=0}^{3}\gamma_\mu p_\mu + \widetilde{m}}{E^2+\phi^2\left|\vec{p}\right|^2 - \widetilde{m}^2} = \frac{\sum_{\mu=0}^{3}\gamma_\mu p_\mu + \widetilde{m}}{m_0^2(p) - \widetilde{m}^2}. \tag{1.16}$$

The factor $m_0^2(p) \in \mathbb{C}_0$ varies not only in α but also in energy and momentum. If $\widetilde{m}$ would be assumed real, it would mean that there is no pole in $G(p)$ on the real p_μ parameter space except for the Euclidean ($\phi^2 = 1$, $\alpha = 0, \pm\pi, \pm 2\pi, \ldots$) and Minkowskian ($\phi^2 = -1$, $\alpha = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$) edge cases.

This raises questions, coming from conventional theory where the propagator of a plane-wave particle would be expected to have a pole at the particle's invariant mass $m_{\mathbb{R}}$. There it would mark the exact momentum and energy associated with the infinite plane wave. Invariant mass $m_{\mathbb{R}}$ is a particle property, just as is α. As such, it seems from Green's function that it is not correct to model the generalized Dirac equation using the real mass property $\widetilde{m} \overset{?}{\equiv} m_{\mathbb{R}}$, but instead the complex mass property $\widetilde{m} \overset{?}{\equiv} \sqrt{m_0^2}$ should be used. Nevertheless, for now the placeholder $\widetilde{m}$ continues to be used until more evidence is gathered.

2. Rutherford scattering of a spin-$\frac{1}{2}$ particle for general α

In order to check for plausibility of the overall approach, and to get a feeling on what challenges may arise when generalizing the calculation, execute a simple quantum calculation in a special case. Observe how the phase α plays into that special case, and learn how this may later need to be handled generally.

Spin-$\frac{1}{2}$ Coulomb scattering (Rutherford scattering) in Born approximation is a simple and well understood calculation that can be done from the Dirac equation with minimal prerequisites. The calculation will clean up the $\alpha = 0$ case in [5], but then also provide the result for general alpha:

- Fix a representation of the Dirac equation and introduce the phase α in the simplest possible way.
- Define some basics (adjoint wave function, conservation of probability, particle propagator).
- Use Born approximation to calculate cross section.

Interpret the result:

- Backscattering, cross section for high energies,
- compare with cross section of "scattering a rock on a Black Hole",
- estimate effects on intergalactic gas distribution,
- estimate momentum transfer $\triangle p$ from scattering high-energy neutrinos on atomic nuclei when traveling through matter.

2.1. **Adjoint wave function and conservation of probability.** For a given volume and time interval without sources, probability must be conserved in the general case. This is a prerequisite for calculating transition properties using perturbation theory methods, but is also a physical principle underlying quantum mechanics in general.

Define a matrix γ_ϕ as:

$$\gamma_\phi := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \phi & 0 \\ 0 & 0 & 0 & \phi \end{pmatrix} = \begin{pmatrix} \sigma_0 & 0 \\ 0 & \phi\sigma_0 \end{pmatrix}, \qquad \gamma_\phi^2 = \gamma_{\phi^2}, \tag{2.1}$$

with

$$\gamma_{\underline{\phi}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \underline{\phi} & 0 \\ 0 & 0 & 0 & \underline{\phi} \end{pmatrix} = \begin{pmatrix} \sigma_0 & 0 \\ 0 & \underline{\phi}\sigma_0 \end{pmatrix}, \qquad \gamma_{\underline{\phi}}^2 = \gamma_{\underline{\phi}^2}, \tag{2.2}$$

accordingly. Therefore

$$\gamma_{\underline{\phi}}\gamma_{\phi} = \gamma_{\phi}\gamma_{\underline{\phi}} = \gamma_{\underline{\phi}^2}\gamma_{\phi^2} = \gamma_{\phi^2}\gamma_{\underline{\phi}^2} \quad = \quad I_4. \tag{2.3}$$

It relates a γ_μ matrix with its $\mathbb{C}$-hermitian transpose, $\overline{\gamma}_\mu^T$:

$$\gamma_\mu = \gamma_{\underline{\phi}^2}\overline{\gamma}_\mu^T\gamma_{\phi^2}, \qquad \overline{\gamma}_\mu^T = \gamma_{\phi^2}\gamma_\mu\gamma_{\underline{\phi}^2}. \tag{2.4}$$

Writing $\psi^\dagger$ for adjoint and ψ^T for transpose of ψ, all $\mathbb{R}^4 \to (\mathbb{C}\times\mathbb{C}_0)^4$, the probability density four-vector $j : \mathbb{R}^4 \to \mathbb{R}^4$ is defined as:

$$j_\mu \quad := \quad \psi^\dagger\gamma_\mu\psi \text{ with } \mu = 0\ldots 3. \tag{2.5}$$

In the absence of sources (charges and fields) probability must be conserved globally:

$$\sum_{\mu=0}^{3}\partial_\mu j_\mu \quad \overset{!}{=} \quad 0. \tag{2.6}$$

This fixes the adjoint as:

$$\psi^\dagger \quad := \quad \overline{\psi}^T\gamma_{\phi^2}. \tag{2.7}$$

Proof. Build the $\mathbb{C}$-hermitian transpose of the Dirac equation, $\sum_{\mu=0}^{3} i\gamma_\mu\partial_\mu\psi = \widetilde{m}\psi$. Conjugation is done in $\mathbb{C}$ since the quantum mechanical probability amplitude is modeled in this subalgebra. The differentials ∂_μ are understood as acting on the wave function $\overline{\psi}^T$, i.e., to the left in this case:

$$\overline{\psi}^T\sum_{\mu=0}^{3}\left(-i\overline{\gamma}_\mu^T\partial_\mu\right) \quad = \quad \overline{\psi}^T\widetilde{m}, \tag{2.8}$$

$$\overline{\psi}^T\sum_{\mu=0}^{3}\left(-i\gamma_{\phi^2}\gamma_\mu\gamma_{\underline{\phi}^2}\partial_\mu\right) \quad = \quad \overline{\psi}^T\widetilde{m}. \tag{2.9}$$

Pull out γ_{ϕ^2} to the left, identify $\widetilde{m} = \gamma_{\phi^2}\widetilde{m}\gamma_{\underline{\phi}^2}$, then multiply with γ_{ϕ^2} from the right and identify the adjoint $\psi^\dagger$:

$$\overline{\psi}^T\gamma_{\phi^2}\sum_{\mu=0}^{3}\left(-i\gamma_\mu\gamma_{\underline{\phi}^2}\partial_\mu\right) \quad = \quad \overline{\psi}^T\gamma_{\phi^2}\widetilde{m}\gamma_{\underline{\phi}^2}, \tag{2.10}$$

$$\overline{\psi}^T\gamma_{\phi^2}\sum_{\mu=0}^{3}\left(-i\gamma_\mu\partial_\mu\right) \quad = \quad \overline{\psi}^T\gamma_{\phi^2}\widetilde{m}, \tag{2.11}$$

$$\psi^\dagger\sum_{\mu=0}^{3}\left(-i\gamma_\mu\partial_\mu\right) \quad = \quad \psi^\dagger\widetilde{m}, \tag{2.12}$$

$$\sum_{\mu=0}^{3}\left(\left(\partial_\mu\psi^\dagger\right)\gamma_\mu\right) \quad = \quad i\widetilde{m}\psi^\dagger. \tag{2.13}$$

The last line only reordered the terms. With this:

$$\sum_{\mu=0}^{3}\partial_\mu j_\mu = \sum_{\mu=0}^{3}\partial_\mu\left(\psi^\dagger\gamma_\mu\psi\right) = \sum_{\mu=0}^{3}\left(\left(\partial_\mu\psi^\dagger\gamma_\mu\right)\psi + \psi^\dagger\left(\gamma_\mu\partial_\mu\psi\right)\right) = i\widetilde{m}\psi^\dagger\psi - i\widetilde{m}\psi^\dagger\psi = 0. \tag{2.14}$$

Probability density is conserved. □

Note that this proof holds regardless of whether $\widetilde{m}$ is real-valued or complex-valued in $\mathbb{C}_0$. The restriction on $\widetilde{m}$ is that it must not be complex-valued in $\mathbb{C}$, since complex conjugation in that space is used for the proof.

The classical adjoint $\psi^\dagger\big|_{\text{QED}}$ is recovered as expected for $\phi^2 = -1$ as

$$\psi^\dagger\big|_{\text{QED}} \quad = \quad \overline{\psi}^T\gamma_0\big|_{\text{QED}} = \overline{\psi}^T\begin{pmatrix}\sigma_0 & 0\\ 0 & -\sigma_0\end{pmatrix}. \tag{2.15}$$

2.2. **Solutions of the free equation of motion.** The free Dirac equation, $\sum_{\mu=0}^{3} i\gamma_\mu \partial_\mu \psi = \widetilde{m}\psi$ (1.2) is explicitly:

$$i\left[\sum_{\mu=0}^{3}\gamma_\mu\partial_\mu\right]\psi = \widetilde{m}\psi, \tag{2.16}$$

$$i\left[\begin{pmatrix}\sigma_0 & 0\\ 0 & -\sigma_0\end{pmatrix}\partial_0 + \sum_{j=1}^{3}\begin{pmatrix}0 & \phi^2\sigma_j\\ \sigma_j & 0\end{pmatrix}\partial_j\right]\psi = \widetilde{m}\psi, \tag{2.17}$$

$$i\left[\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\partial_0 + \begin{pmatrix}0&0&0&\phi^2\\0&0&\phi^2&0\\0&1&0&0\\1&0&0&0\end{pmatrix}\partial_1 + \right. \tag{2.18}$$

$$\left.\begin{pmatrix}0&0&0&-i\phi^2\\0&0&i\phi^2&0\\0&-i&0&0\\i&0&0&0\end{pmatrix}\partial_2 + \begin{pmatrix}0&0&\phi^2&0\\0&0&0&-\phi^2\\1&0&0&0\\0&-1&0&0\end{pmatrix}\partial_3\right]\psi = \widetilde{m}\psi,$$

$$\begin{pmatrix} i\partial_0 & 0 & \phi^2(i\partial_3) & \phi^2(i\partial_1+\partial_2)\\ 0 & i\partial_0 & \phi^2(i\partial_1-\partial_2) & \phi^2(-i\partial_3)\\ i\partial_3 & i\partial_1+\partial_2 & -i\partial_0 & 0\\ i\partial_1-\partial_2 & -i\partial_3 & 0 & -i\partial_0\end{pmatrix}\psi = \widetilde{m}\psi. \tag{2.19}$$

Finding eigenfunctions and eigenvalues requires an energy-momentum-mass relation, which is $|E|^2 - |\vec{p}|^2 = m^2_{\text{Mink}}$ in the classical case, and $|E|^2 + |\vec{p}|^2 = m^2_{\text{Eucl}}$ in the 4D Euclidean case ([2]). This obviously has to be generalized now due to the phase α contained in ϕ^2. Equation (1.9) has the consistent generalization,

$$m_0^2 = E^2 + \phi^2 |\vec{p}|^2 . \tag{2.20}$$

This is not consistent any more with a real-valued $\widetilde{m}|_{\text{QED}} \equiv m_{\mathbb{R}} \in \mathbb{R}$ as in the classical case. Eigenfunctions of this linear differential equation - to be found - must contain an exponential function part $\sim \exp(f(p))$, as well as a vector part. The vector part will contain terms of $\vec{p}$ and E that multiply with additional terms $\vec{p}$ and E from the differential on the exponential function part. In order for $\widetilde{m}$ to be real-valued, all factors ϕ^2 would have to cancel out directly from these differentials. However, since all momentum differentials in (2.19) appear both with and without factors ϕ^2, this is impossible in principle. Therefore, $\widetilde{m}$ must be complex-valued in $\mathbb{C}_0$ and the placeholder $\widetilde{m}$ is determined to be m_0 (and not $m_{\mathbb{R}}$) going forward:

$$\widetilde{m} := m_0, \tag{2.21}$$

$$i\left[\sum_{\mu=0}^{3}\gamma_\mu\partial_\mu\right]\psi = m_0\psi. \tag{2.22}$$

There is an underlying choice that is made to arrive at this identification: The generalized Dirac equation is to remain as similar as possible to the classical case, with as little as needed modification in the formulation as possible.

With this, eigenfunctions $\widetilde{\Psi}^{\pm}_{1/2}$ can be found:

$$\widetilde{\Psi}_1^+ := \exp i(\vec{p}\vec{x} - Et)\begin{pmatrix}1\\0\\-p_3/(m_0+E)\\(-p_1-ip_2)/(m_0+E)\end{pmatrix}, \qquad \widetilde{\Psi}_1^- := \exp i(\vec{p}\vec{x} - Et)\begin{pmatrix}0\\1\\(-p_1+ip_2)/(m_0+E)\\p_3/(m_0+E)\end{pmatrix}, \tag{2.23}$$

$$\widetilde{\Psi}_2^+ := \exp i(\vec{p}\vec{x} + Et)\begin{pmatrix}-\phi^2 p_3/(m_0+E)\\ \phi^2(-p_1-ip_2)/(m_0+E)\\1\\0\end{pmatrix}, \qquad \widetilde{\Psi}_2^- := \exp i(\vec{p}\vec{x} + Et)\begin{pmatrix}\phi^2(-p_1+ip_2)/(m_0+E)\\ \phi^2 p_3/(m_0+E)\\0\\1\end{pmatrix}. \tag{2.24}$$

In comparison to the classical solutions ($\phi^2 = -1$) the $\widetilde{\Psi}^{\pm}_{1/2}$ can be identified as (anti)particle (1/2) plane waves with spin up (+) or down (−). Here in the general case, m_0 varies in $\mathbb{C}_0$ for all solutions, and there is a phase ϕ^2 in the vector parts of the eigenfunctions between particles and antiparticles.

Using a spin vector $\chi^{\pm}$ with

$$\chi^+ := \begin{pmatrix}1\\0\end{pmatrix}, \qquad \chi^- := \begin{pmatrix}0\\1\end{pmatrix}, \tag{2.25}$$

the eigenfunctions can be expressed using Pauli spinors σ_μ and $\vec{\sigma} := (\sigma_1, \sigma_2, \sigma_3)$ as:

$$\widetilde{\Psi}_1^{\pm} := \exp i(\vec{p}\vec{x} - Et)\begin{pmatrix}\sigma_0\\ -\vec{\sigma}\vec{p}/(m_0+E)\end{pmatrix}\chi^{\pm}, \tag{2.26}$$

$$\widetilde{\Psi}_2^{\pm} := \exp i(\vec{p}\vec{x} + Et)\begin{pmatrix}-\phi^2\vec{\sigma}\vec{p}/(m_0+E)\\ \sigma_0\end{pmatrix}\chi^{\pm}. \tag{2.27}$$

2.3. **Normalizing the plane-wave eigenfunctions.** Norms of numbers $a \in \mathbb{C} \times \mathbb{C}_0$ are defined in $\mathbb{C}$, $\mathbb{C}_0$, and $\mathbb{C} \times \mathbb{C}_0$ as:

$$|a|^2 := a\bar{a} \in \mathbb{C}_0, \qquad |a|_0^2 := a\underline{a} \in \mathbb{C}, \qquad \|a\|^4 := a\bar{a}\underline{a}\underline{\bar{a}} \in \mathbb{R}. \tag{2.28}$$

The term "norm" is used loosely, in the sense that the composition property is conserved for any $a, b \in \mathbb{C} \times \mathbb{C}_0$:

$$|ab|^2 = |a|^2 |b|^2, \qquad |ab|_0^2 = |a|_0^2 |b|_0^2, \qquad \|ab\|^4 = \|a\|^4 \|b\|^4. \tag{2.29}$$

Norms are, however, not positive definite or point separating. Note that when taking the square (or fourth) root of these expressions it has to be made clear which space the result is to be in ($\mathbb{C}$, $\mathbb{C}_0$, or $\mathbb{C} \times \mathbb{C}_0$).

Since the generalized Dirac equation uses an eigenvalue $m_0 \in \mathbb{C}_0$, the calculation of the scattering cross section will be executed in the $\mathbb{C}$ subalgebra only, i.e., in the same manner as in the classical case. This is possible since all dynamic variables act in this subalgebra, and the only variable in $\mathbb{C}_0$ is α itself which is constant in space and time (and therewith in p). Only at the very end, when asking for probabilities, will the $\mathbb{C}_0$ norm be taken, to obtain a real value.

The $\widetilde{\Psi}^{\pm}_{1/2}$ above are obtained by fixing components $(1,0)$ and $(0,1)$ in their vector parts. They are not yet normed to conserve probability in field-free space (and time). The normed eigenfunctions $\hat{\Psi}^{\pm}_{1/2}$ satisfy

$$\left(\hat{\Psi}^{\pm}_{1/2}\right)^{\dagger} \hat{\Psi}^{\pm}_{1/2} \stackrel{!}{=} 1. \tag{2.30}$$

They differ from the $\widetilde{\Psi}^{\pm}_{1/2}$ only by a constant factor $N_{1/2} \in \mathbb{C}_0$,

$$\hat{\Psi}^{\pm}_{1/2} := N_{1/2}\widetilde{\Psi}^{\pm}_{1/2}. \tag{2.31}$$

For $\widetilde{\Psi}^{+}_{1}$ there is:

$$\left(\hat{\Psi}^{+}_{1}\right)^{\dagger} \hat{\Psi}^{+}_{1} = \overline{\left(\hat{\Psi}^{+}_{1}\right)}^{T} \gamma_{\phi^2} \hat{\Psi}^{+}_{1} = \overline{N_1}\,\overline{\left(\widetilde{\Psi}^{+}_{1}\right)}^{T} \gamma_{\phi^2} N_1 \widetilde{\Psi}^{+}_{1} \tag{2.32}$$

$$= |N_1|^2 \left(1, 0, \frac{-p_3}{m_0+E}, \frac{-p_1+ip_2}{m_0+E}\right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \phi^2 & 0 \\ 0 & 0 & 0 & \phi^2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -p_3/(m_0+E) \\ (-p_1-ip_2)/(m_0+E) \end{pmatrix} \tag{2.33}$$

$$= |N_1|^2 \left(1 + \frac{\phi^2 p_3^2}{(m_0+E)^2} + \frac{\phi^2 (p_1^2+p_2^2)}{(m_0+E)^2}\right) \tag{2.34}$$

$$= |N_1|^2 \left(1 + \frac{\phi^2 |\vec{p}|^2}{(m_0+E)^2}\right). \tag{2.35}$$

Using $m_0^2 = E^2 + \phi^2 |\vec{p}|^2$ this becomes:

$$\left(\hat{\Psi}^{+}_{1}\right)^{\dagger} \hat{\Psi}^{+}_{1} = |N_1|^2 \frac{(m_0+E)^2 + \phi^2 |\vec{p}|^2}{(m_0+E)^2} = |N_1|^2 \frac{m_0^2 + 2m_0E + E^2 + \phi^2 |\vec{p}|^2}{(m_0+E)^2} \tag{2.36}$$

$$= |N_1|^2 \frac{m_0^2 + 2m_0E + m_0^2}{(m_0+E)^2} = |N_1|^2 \frac{2m_0 (m_0+E)}{(m_0+E)^2} \tag{2.37}$$

$$= |N_1|^2 \frac{2m_0}{m_0+E}. \tag{2.38}$$

The normalizing factor $N_1 \in \mathbb{C}_0$ therefore differs from the classical case only by generalizing real mass $m_{\mathbb{R}}$ to m_0:

$$|N_1|^2 = \frac{m_0+E}{2m_0} \in \mathbb{C}_0, \qquad N_1 = \sqrt{\frac{m_0+E}{2m_0}}. \tag{2.39}$$

The square root is to be taken in $\mathbb{C}_0$. The identical calculation can be done for the $\widetilde{\Psi}^{-}_{1}$.

For the $\widetilde{\Psi}^{+}_{2}$ we have:

$$\left(\hat{\Psi}^{+}_{2}\right)^{\dagger} \hat{\Psi}^{+}_{2} = |N_2|^2 \left(\frac{-\phi^2 p_3}{m_0+E}, \frac{-\phi^2 p_1 + i\phi^2 p_2}{m_0+E}, 1, 0\right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \phi^2 & 0 \\ 0 & 0 & 0 & \phi^2 \end{pmatrix} \begin{pmatrix} -\phi^2 p_3/(m_0+E) \\ \phi^2 (-p_1-ip_2)/(m_0+E) \\ 1 \\ 0 \end{pmatrix} \tag{2.40}$$

$$= \phi^2 |N_2|^2 \left(\frac{\phi^2 p_3^2}{(m_0+E)^2} + \frac{\phi^2 (p_1^2+p_2^2)}{(m_0+E)^2} + 1\right) \tag{2.41}$$

$$= \phi^2 |N_2|^2 \frac{\phi^2 |\vec{p}|^2 + (m_0+E)^2}{(m_0+E)^2} \tag{2.42}$$

$$= \phi^2 |N_2|^2 \frac{2m_0}{m_0+E}. \tag{2.43}$$

Since $\phi\underline{\phi} = 1$ this yields:

$$|N_2|^2 = \underline{\phi}^2 \frac{m_0 + E}{2m_0} \in \mathbb{C}_0, \qquad N_2 = \underline{\phi}\sqrt{\frac{m_0 + E}{2m_0}}. \tag{2.44}$$

The constant N_2 is rotated by a phase $\underline{\phi}$ as compared to N_1,r

$$N_2 \;=\; \underline{\phi} N_1. \tag{2.45}$$

This will be of no consequence in this paper since particles and antiparticles won't change into one another during elastic scattering. For future work that investigates interactions and transitions involving both particles and antiparticles, e.g. fermion pair production and annihilation, this phase may affect the measurement prediction.

You could of course multiply $\underline{\phi}^2$ into the eigenfunctions $\widetilde{\Psi}_2^\pm$ themselves:

$$\widetilde{\Psi}_1^\pm \;=\; \exp i\left(\vec{p}\vec{x} - Et\right) \begin{pmatrix} \sigma_0 \\ -\vec{\sigma}\vec{p}/\left(m_0 + E\right) \end{pmatrix} \chi^\pm, \tag{2.46}$$

$$\widetilde{\Psi}_2^\pm \underline{\phi}^2 \;=\; \exp i\left(\vec{p}\vec{x} + Et\right) \begin{pmatrix} -\vec{\sigma}\vec{p}/\left(m_0 + E\right) \\ \underline{\phi}^2 \sigma_0 \end{pmatrix} \chi^\pm. \tag{2.47}$$

While this would make the momentum parts $-\vec{\sigma}\vec{p}/\left(m_0 + E\right)$ symmetric between the $\widetilde{\Psi}_1^\pm$ and $\widetilde{\Psi}_2^\pm \underline{\phi}^2$, the σ_0 parts would become asymmetric. Since there's no real gain for the purpose of this paper, the definitions are left unchanged.

The normalized component of the eigenfunctions that depends on the particle's spin and type, $u_{1/2}^\pm$, is defined as:

$$u_1^\pm \;:=\; N_1 \begin{pmatrix} \sigma_0 \\ -\vec{\sigma}\vec{p}/\left(m_0 + E\right) \end{pmatrix} \chi^\pm = \sqrt{\frac{m_0 + E}{2m_0}} \begin{pmatrix} \sigma_0 \\ -\vec{\sigma}\vec{p}/\left(m_0 + E\right) \end{pmatrix} \chi^\pm, \tag{2.48}$$

$$u_1^+ = \sqrt{\frac{m_0 + E}{2m_0}} \begin{pmatrix} 1 \\ 0 \\ -p_3/\left(m_0 + E\right) \\ \left(-p_1 - ip_2\right)/\left(m_0 + E\right) \end{pmatrix}, \qquad u_1^- = \sqrt{\frac{m_0 + E}{2m_0}} \begin{pmatrix} 0 \\ 1 \\ \left(-p_1 + ip_2\right)/\left(m_0 + E\right) \\ p_3/\left(m_0 + E\right) \end{pmatrix},$$

$$u_2^\pm \;:=\; N_2 \begin{pmatrix} -\phi^2 \vec{\sigma}\vec{p}/\left(m_0 + E\right) \\ \sigma_0 \end{pmatrix} \chi^\pm = \sqrt{\frac{m_0 + E}{2m_0}} \begin{pmatrix} -\phi\vec{\sigma}\vec{p}/\left(m_0 + E\right) \\ \underline{\phi}\sigma_0 \end{pmatrix} \chi^\pm, \tag{2.49}$$

$$u_2^+ = \sqrt{\frac{m_0 + E}{2m_0}} \begin{pmatrix} -\phi p_3/\left(m_0 + E\right) \\ \phi\left(-p_1 - ip_2\right)/\left(m_0 + E\right) \\ \underline{\phi} \\ 0 \end{pmatrix}, \qquad u_2^- = \sqrt{\frac{m_0 + E}{2m_0}} \begin{pmatrix} \phi\left(-p_1 + ip_2\right)/\left(m_0 + E\right) \\ \phi p_3/\left(m_0 + E\right) \\ 0 \\ \underline{\phi} \end{pmatrix}.$$

This simply leaves out the oscillating plane-wave component of the eigenfunctions:

$$\hat{\Psi}_k^\pm \;=\; u_k^\pm \exp i\left(\vec{p}\vec{x} + (-1)^k Et\right) \qquad \text{with } k = 1, 2. \tag{2.50}$$

2.4. **Normalizing to invariant volume.** Following textbook calculation of the elastic (Rutherford) scattering cross section on a fixed target in Born approximation, the wave functions Ψ of the incoming (ini) and outgoing (fin) particle will be normalized to a small invariant volume V_0 in the reference frame of the target. Per equation (1.14) the invariant volume V_0 is

$$V_0 \;=\; \frac{VE}{m_0} = \frac{V}{\sqrt{1 + \phi^2 \left|\vec{v}\right|^2}}, \tag{2.51}$$

and the such normalized $\Psi_k^\pm$ are:

$$\Psi_k^\pm \;:=\; \sqrt{\frac{m_0}{VE}} \hat{\Psi}_k^\pm \tag{2.52}$$

$$\;=\; \sqrt{\frac{m_0}{VE}} u_k^\pm \exp i\left(\vec{p}\vec{x} + (-1)^k Et\right). \tag{2.53}$$

This satisfies

$$\left(\Psi_k^\pm\right)^\dagger \Psi_k^\pm \;=\; \frac{1}{V_0}. \tag{2.54}$$

2.5. **Coulomb-type field of a point charge.** In the classical $\phi^2 = -1$ case, the electromagnetic field is introduced by requiring invariance of the Dirac equation with field under a simple $\exp\left(iq\chi\right)$ phase. There, q is the electric charge of the particle under the influence of the field, and space-time derivatives of χ are identified as the electromagnetic potentials A_μ acting on the particle proportionally to q with inertia $m_\mathbb{R}$.

This is now generalized analogously, writing placeholder symbols $\widetilde{q}$ and $\widetilde{\chi}$ for now until it is clarified exactly which space they're in:

(2.55) $$\widetilde{A}_\mu := \frac{\partial \widetilde{\chi}}{\partial x_\mu}, \qquad \psi' := e^{i\widetilde{q}\widetilde{\chi}}\psi,$$

(2.56) $$\sum_{\mu=0}^{3} \gamma_\mu \left(i\partial_\mu\right)\psi' = m_0\psi' \qquad \longrightarrow \qquad e^{i\widetilde{q}\widetilde{\chi}} \sum_{\mu=0}^{3} \gamma_\mu \left(i\partial_\mu - \widetilde{q}\widetilde{A}_\mu\right)\psi = m_0\psi'.$$

In the general case there is

(2.57) $$\widetilde{q}\widetilde{A}_\mu \in \mathbb{C}_0,$$

(2.58) $$\sum_{\mu=0}^{3} \gamma_\mu \left(i\partial_\mu + \widetilde{q}\widetilde{A}_\mu\right)\psi = m_0\psi.$$

Equation (2.58) is the generalized Dirac equation, invariant under $\mathrm{U}(1)$ gauge (in $\mathbb{C}$).

This raises the question where exactly the phase in $\mathbb{C}_0$ originates from: Is one of the $\left\{\widetilde{q}, \widetilde{A}_\mu\right\}$ real or are both complex in $\mathbb{C}_0$? By analogy, the purely electromagnetic and gravitational edge cases have $\{m_{\mathrm{Eucl}}, m_{\mathrm{Mink}}\} \in \mathbb{R}$, and we would expect in the gravitational case for the charge to become its mass, $\widetilde{q}_{\mathrm{Eucl}} \equiv m_{\mathrm{Eucl}}$. In the electromagnetic case we expect real charges as well, making $\{\widetilde{q}_{\mathrm{Eucl}}, \widetilde{q}_{\mathrm{Mink}}\} \in \mathbb{R}$. Both edge cases appear, however, unphysical: Purely electromagnetic interaction would assume a particle that is charged, however, doesn't interact gravitationally; and purely gravitational interaction would assume a particle or field that has no kinetic component bound to Minkowskian spacetime. Addressing these concerns is left for later.

Without needing to make speculations on the dynamics behind the $\widetilde{A}_\mu$, a static Coulomb potential of a charge $\widetilde{Q}$ is now modeled as:

(2.59) $$\widetilde{A}_0 := -\frac{\widetilde{Q}}{|\vec{x}|}, \qquad \widetilde{A}_j = 0 \text{ otherwise},$$

(2.60) $$\Longrightarrow \qquad \sum_{\mu=0}^{3} \gamma_\mu \widetilde{q}\widetilde{A}_\mu = -\gamma_0 \frac{\widetilde{q}\widetilde{Q}}{|\vec{x}|}.$$

The generalized Dirac equation for a spin-1/2 particle in a static Coulomb potential then is:

(2.61) $$\left(\sum_{\mu=0}^{3} i\gamma_\mu\partial_\mu - \gamma_0\frac{\widetilde{q}\widetilde{Q}}{|\vec{x}|}\right)\psi = m_0\psi'.$$

2.6. **Lowest-order transition matrix element in Born approximation.** For scattering on the fixed point target, initial (incoming) and final (outgoing) wave functions $\Psi^{\pm}_{k,\mathrm{ini}}$ and $\Psi^{\pm}_{k,\mathrm{fin}}$ of the spin-1/2 particle are assumed plane waves. Only the lowest-order transition matrix element S_{fi} is calculated. Using this customary approximation, results can then be compared qualitatively with conventional QED results. Some quantitative estimates will also be possible.

(2.62) $$\Psi^{\pm}_{k,\mathrm{ini}} := \sqrt{\frac{m_0}{VE}} u^{\pm}_{k,\mathrm{ini}} \exp i\left(\vec{p}\vec{x} + (-1)^k Et\right), \qquad \Psi^{\pm}_{k,\mathrm{fin}} := \sqrt{\frac{m_0}{VE}} u^{\pm}_{k,\mathrm{fin}} \exp i\left(\vec{p}\vec{x} + (-1)^k Et\right),$$

For now omitting the annotations $\pm$ and k for readability:

(2.63) $$S_{\mathrm{fi}} = \langle\Psi_{\mathrm{fin}}|\, S\, |\Psi_{\mathrm{ini}}\rangle = i\int d^4x \Psi^{\dagger}_{\mathrm{fin}} \left(\widetilde{q}\sum_{\mu=0}^{3}\gamma_\mu A_\mu\right)\Psi_{\mathrm{ini}}$$

(2.64) $$= i\int d^4x \Psi^{\dagger}_{\mathrm{fin}} \left(-\widetilde{q}\gamma_0\frac{\widetilde{Q}}{|\vec{x}|}\right)\Psi_{\mathrm{ini}}$$

(2.65) $$= -i\widetilde{q}\widetilde{Q}\int d^4x \left(\sqrt{\frac{m_0}{VE_{\mathrm{fin}}}} u_{\mathrm{fin}} \exp i\left(\vec{p}_{\mathrm{fin}}\vec{x} + (-1)^k E_{\mathrm{fin}}t\right)\right)^{\dagger} \left(\gamma_0\frac{1}{|\vec{x}|}\right) \left(\sqrt{\frac{m_0}{VE_{\mathrm{ini}}}} u_{\mathrm{ini}} \exp i\left(\vec{p}_{\mathrm{ini}}\vec{x} + (-1)^k E_{\mathrm{ini}}t\right)\right)$$

(2.66) $$= -i\frac{\widetilde{q}\widetilde{Q}m_0}{V\sqrt{E_{\mathrm{fin}}E_{\mathrm{ini}}}} u^{\dagger}_{\mathrm{fin}}\gamma_0 u_{\mathrm{ini}} \int d^4x \frac{\exp\left[i\sum_{\nu=0}^{3}\left(p_{\nu,\mathrm{fin}} - p_{\nu,\mathrm{ini}}\right)\right]}{|\vec{x}|}.$$

As compared to the classical case, the differences are in the constants $\widetilde{q}$, $\widetilde{Q}$, and m_0 (all valued in $\mathbb{C}_0$), as well as the $u^{\pm}_k$ and their adjoint $\left(u^{\pm}_k\right)^{\dagger} := \overline{u^{\pm}_k}^T \gamma_{\phi^2}$ which contain constant terms in $\mathbb{C}_0$ as well. Here, "constant" means independent of dynamic variables $(\vec{x}, t)$ and properties $(\vec{p}, E)$.

When calculating the transition probability dW for a single particle into a particular state dN (here, into a volume V and momentum interval d^3p_{fin}),

(2.67) $$dW = \|S_{\mathrm{fi}}\|^2\, dN,$$

the norm in $\mathbb{C} \times \mathbb{C}_0$ with $\|a\|^4 := a\bar{a}\underline{a}\underline{\bar{a}}$ will be used since it is guaranteed real-valued. Other than this intuitive generalization, standard methods can be followed for calculating the spin-independent part $d\widetilde{\sigma}_{\text{class}}/d\Omega$ of the cross section $d\widetilde{\sigma}$ into an angle element $d\Omega$:

$$\frac{d\widetilde{\sigma}}{d\Omega} = \frac{d\widetilde{\sigma}_{\text{class}}}{d\Omega} \frac{d\widetilde{\sigma}_{\text{spin}}}{d\Omega}, \tag{2.68}$$

$$\frac{d\widetilde{\sigma}_{\text{class}}}{d\Omega} = \frac{\left|\widetilde{q}\widetilde{Q}m_0\right|^2}{4\left|\vec{p}\right|^4 \sin^4 \frac{\theta}{2}}. \tag{2.69}$$

In order to obtain the real-valued $d\sigma$ at the end, we'll simply take its absolute again:

$$\frac{d\sigma}{d\Omega} = \left\| \frac{d\widetilde{\sigma}}{d\Omega} \right\| \equiv \left| \frac{d\widetilde{\sigma}}{d\Omega} \right|_0 . \tag{2.70}$$

2.7. **Spin contribution.** The spin contribution to the cross section (2.68) is

$$\frac{d\widetilde{\sigma}_{\text{spin}}}{d\Omega} = \sum_{k,\pm} \left| u_{\text{fin}}^{\dagger} \gamma_0 u_{\text{ini}} \right|^2 . \tag{2.71}$$

The symbols k and $\pm$ indicate summation over all possible transitions from the incoming to the outgoing wave.

2.7.1. *Particle / antiparticle transition, spin flip.* In the classical case the transition matrix elements for changing from a particle into an antiparticle, and vice versa, are trivially zero. Here we have additional factors ϕ and $\underline{\phi}$ to be tested. Using definitions for the $u_k^{\pm}$ we have:

$$\left(u_1^+\right)^\dagger \gamma_0 u_2^+ = \frac{m_0+E}{2m_0}\left(1, 0, \frac{-p_3}{m_0+E}, \frac{-p_1+ip_2}{m_0+E}\right)\gamma_{\phi^2}\gamma_0 \begin{pmatrix} -\phi p_3/\left(m_0+E\right) \\ \phi\left(-p_1-ip_2\right)/\left(m_0+E\right) \\ \underline{\phi} \\ 0 \end{pmatrix} = \phi \frac{-p_3+p_3}{2m_0} = 0, \tag{2.72}$$

$$\left(u_1^+\right)^\dagger \gamma_0 u_2^- = \frac{m_0+E}{2m_0}\left(1, 0, \frac{-p_3}{m_0+E}, \frac{-p_1+ip_2}{m_0+E}\right)\gamma_{\phi^2}\gamma_0 \begin{pmatrix} \phi\left(-p_1+ip_2\right)/\left(m_0+E\right) \\ \phi p_3/\left(m_0+E\right) \\ 0 \\ \underline{\phi} \end{pmatrix} = \phi \frac{-p_1+ip_2+p_1-ip_2}{2m_0} = 0. \tag{2.73}$$

The other six cases of the $u_{\text{fin}}^{\dagger}\gamma_0 u_{\text{ini}}$ where $k_{\text{fin}} \neq k_{\text{ini}}$ are symmetric through reordering of the vector components, as well as swapping factors in the commutative product. A particle cannot change into an antiparticle through elastic scattering, and vice versa.

Spin flip is also excluded:

$$\left(u_1^+\right)^\dagger \gamma_0 u_1^+ = \frac{m_0+E}{2m_0}\left(1, 0, \frac{-p_3}{m_0+E}, \frac{-p_1+ip_2}{m_0+E}\right)\gamma_{\phi^2}\gamma_0 \begin{pmatrix} 0 \\ 1 \\ \left(-p_1+ip_2\right)/\left(m_0+E\right) \\ p_3/\left(m_0+E\right) \end{pmatrix} = 0, \tag{2.74}$$

$$\left(u_2^+\right)^\dagger \gamma_0 u_2^+ = \frac{m_0+E}{2m_0}\left(\frac{-\phi p_3}{m_0+E}, \frac{\phi\left(-p_1+ip_2\right)}{m_0+E}, \underline{\phi}, 0\right)\gamma_{\phi^2}\gamma_0 \begin{pmatrix} \phi\left(-p_1+ip_2\right)/\left(m_0+E\right) \\ \phi p_3/\left(m_0+E\right) \\ 0 \\ \underline{\phi} \end{pmatrix} = 0. \tag{2.75}$$

2.7.2. *Spin and particle type remains unchanged.* Interaction of the particle with the central $\widetilde{A} = \left(-\widetilde{Q}/\left|\vec{x}\right|, 0, 0, 0\right)$ potential may only change the particle's momentum distribution across different angles, but not its spin or type. The only spin contribution therefore is:

$$\frac{d\widetilde{\sigma}_{\text{spin}}}{d\Omega} = \left| \left(u_{k,\text{fin}}^{\pm}\right)^\dagger \gamma_0 u_{k,\text{ini}}^{\pm} \right|^2 \quad \text{for any } k, \pm \text{ unchanged.} \tag{2.76}$$

The calculation is symmetric for particles and antiparticles, for spin up and spin down, as well as rotational in space around the axis of incoming momentum respective to the target. It is therefore calculated for a particle ($k = 1$) with spin up ($+$) in the

(x_1, x_2) plane ($p_{3,\text{ini}} = p_{3,\text{fin}} = 0$), assuming energy conservation ($E_{\text{fin}} = E_{\text{ini}} \equiv E$):

$$\left(u^+_{1,\text{fin}}\right)^\dagger \gamma_0 u^+_{1,\text{ini}} = \frac{m_0+E}{2m_0}\left(1,0,0,\frac{-p_{1,\text{fin}}+ip_{2,\text{fin}}}{m_0+E}\right)\gamma_{\phi^2}\gamma_0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ \left(-p_{1,\text{ini}}-ip_{2,\text{ini}}\right)/\left(m_0+E\right) \end{pmatrix} \tag{2.77}$$

$$= \frac{m_0+E}{2m_0}\left(1+\frac{\left(-p_{1,\text{fin}}+ip_{2,\text{fin}}\right)\phi^2\left(p_{1,\text{ini}}+ip_{2,\text{ini}}\right)}{\left(m_0+E\right)^2}\right) \tag{2.78}$$

$$= \frac{\left(m_0+E\right)^2-\phi^2\left|\vec{p}\right|^2\left(\cos\theta-i\sin\theta\right)}{2m_0\left(m_0+E\right)}. \tag{2.79}$$

The last line expressed the difference between initial and final momentum in terms of scattering angle θ, while taking advantage of $p_3 = 0$:

$$p_{1,\text{fin}}p_{1,\text{ini}}+p_{2,\text{fin}}p_{2,\text{ini}} \equiv \vec{p}_{\text{fin}}\vec{p}_{\text{ini}} = \left|\vec{p}\right|^2\cos\theta, \tag{2.80}$$

$$p_{1,\text{fin}}p_{2,\text{ini}}-p_{2,\text{fin}}p_{1,\text{ini}} \equiv \vec{p}_{\text{fin}}\times\vec{p}_{\text{ini}} = -\left|\vec{p}\right|^2\sin\theta. \tag{2.81}$$

Using the identities

$$\cos^2\theta = 1-\sin^2\theta, \tag{2.82}$$

$$\cos\theta = 1-2\sin^2\frac{\theta}{2}, \tag{2.83}$$

there is:

$$\frac{d\widetilde{\sigma}_{\text{spin}}}{d\Omega} = \left|\left(u^+_{1,\text{fin}}\right)^\dagger \gamma_0 u^+_{1,\text{ini}}\right|^2 \tag{2.84}$$

$$= \left|\frac{\left(m_0+E\right)^2-\phi^2\left|\vec{p}\right|^2\left(\cos\theta-i\sin\theta\right)}{2m_0\left(m_0+E\right)}\right|^2 \tag{2.85}$$

$$= \frac{\left(\left(m_0+E\right)^2-\phi^2\left|\vec{p}\right|^2\cos\theta\right)^2+\left(\phi^2\left|\vec{p}\right|^2\sin\theta\right)^2}{\left(2m_0\left(m_0+E\right)\right)^2} \tag{2.86}$$

$$= \frac{\left(m_0+E\right)^4-2\left(m_0+E\right)^2\phi^2\left|\vec{p}\right|^2\cos\theta+\phi^4\left|\vec{p}\right|^4\cos^2\theta+\phi^4\left|\vec{p}\right|^4\sin^2\theta}{\left(2m_0\left(m_0+E\right)\right)^2} \tag{2.87}$$

...using $\cos^2\theta = 1-\sin^2\theta$ ...

$$= \frac{\left(m_0+E\right)^4-2\left(m_0+E\right)^2\phi^2\left|\vec{p}\right|^2\cos\theta+\phi^4\left|\vec{p}\right|^4}{\left(2m_0\left(m_0+E\right)\right)^2} \tag{2.88}$$

...using $\cos\theta = 1-2\sin^2\frac{\theta}{2}$ ...

$$= \frac{\left(m_0+E\right)^4-2\left(m_0+E\right)^2\phi^2\left|\vec{p}\right|^2+\phi^4\left|\vec{p}\right|^4+4\left(m_0+E\right)^2\phi^2\left|\vec{p}\right|^2\sin^2\frac{\theta}{2}}{\left(2m_0\left(m_0+E\right)\right)^2} \tag{2.89}$$

$$= \frac{\left(\left(m_0+E\right)^2-\phi^2\left|\vec{p}\right|^2\right)^2}{\left(2m_0\left(m_0+E\right)\right)^2}+\frac{\phi^2\left|\vec{p}\right|^2\sin^2\frac{\theta}{2}}{m_0^2} \tag{2.90}$$

$$= \frac{\left(m_0^2+2m_0E+E^2-\phi^2\left|\vec{p}\right|^2\right)^2}{\left(2m_0\left(m_0+E\right)\right)^2}+\frac{\phi^2\left|\vec{p}\right|^2\sin^2\frac{\theta}{2}}{m_0^2} \tag{2.91}$$

...using $m_0^2 = E^2+\phi^2\left|\vec{p}\right|^2$ ...

$$= \frac{\left(2E\left(m_0+E\right)\right)^2}{\left(2m_0\left(m_0+E\right)\right)^2}+\frac{\phi^2\left|\vec{p}\right|^2\sin^2\frac{\theta}{2}}{m_0^2} \tag{2.92}$$

$$= \frac{E^2}{m_0^2}+\phi^2\frac{\left|\vec{p}\right|^2\sin^2\frac{\theta}{2}}{m_0^2} \tag{2.93}$$

...using $\dfrac{E^2}{m_0^2} = \dfrac{1}{1+\phi^2\left|\vec{v}\right|^2}$, $\quad \dfrac{\left|\vec{p}\right|^2}{E^2} = \left|\vec{v}\right|^2$ ...

$$= \frac{1+\phi^2\left|\vec{v}\right|^2\sin^2\frac{\theta}{2}}{1+\phi^2\left|\vec{v}\right|^2}. \tag{2.94}$$

This correctly recovers the classical case for $\phi^2\big|_{\text{QED}} = -1$.

2.8. **Result and comparison with the classical case.** Putting the (semi-)classical and spin contributions to the scattering cross section together, the result is:

$$\frac{d\widetilde{\sigma}}{d\Omega} = \frac{d\widetilde{\sigma}_{\text{class}}}{d\Omega}\frac{d\widetilde{\sigma}_{\text{spin}}}{d\Omega} \tag{2.95}$$

$$\text{with } \frac{d\widetilde{\sigma}_{\text{class}}}{d\Omega} = \frac{\left|\widetilde{q}\widetilde{Q}m_0\right|^2}{4\left|\vec{p}\right|^4\sin^4\frac{\theta}{2}}, \qquad \frac{d\widetilde{\sigma}_{\text{spin}}}{d\Omega} = \frac{1}{m_0^2}\left(E^2+\phi^2\left|\vec{p}\right|^2\sin^2\frac{\theta}{2}\right) = \frac{1+\phi^2\left|\vec{v}\right|^2\sin^2\frac{\theta}{2}}{1+\phi^2\left|\vec{v}\right|^2}, \tag{2.96}$$

$$\implies \frac{d\widetilde{\sigma}}{d\Omega} = \frac{\left|\widetilde{q}\widetilde{Q}\right|^2}{4\left|\vec{p}\right|^4\sin^4\frac{\theta}{2}}\left(E^2+\phi^2\left|\vec{p}\right|^2\sin^2\frac{\theta}{2}\right) = \frac{\left|\widetilde{q}\widetilde{Q}m_0\right|^2}{4\left|\vec{p}\right|^4\sin^4\frac{\theta}{2}}\cdot\frac{1+\phi^2\left|\vec{v}\right|^2\sin^2\frac{\theta}{2}}{1+\phi^2\left|\vec{v}\right|^2}, \tag{2.97}$$

$$\frac{d\sigma}{d\Omega} = \left\|\frac{d\widetilde{\sigma}}{d\Omega}\right\| \equiv \left|\frac{d\widetilde{\sigma}}{d\Omega}\right|_0. \tag{2.98}$$

As compared to the classical result,

$$\left.\frac{d\sigma}{d\Omega}\right|_{\text{QED}} = \frac{\left|qQ\right|^2}{4\left|\vec{p}\right|^4\sin^4\frac{\theta}{2}}\left(E^2-\left|\vec{p}\right|^2\sin^2\frac{\theta}{2}\right) \qquad \text{(for general } E, \left|\vec{p}\right|\text{)} \tag{2.99}$$

$$= \frac{\left|qQm\right|^2}{4\left|\vec{p}\right|^4\sin^4\frac{\theta}{2}}\cdot\frac{1-\left|\vec{v}\right|^2\sin^2\frac{\theta}{2}}{1-\left|\vec{v}\right|^2} \qquad \text{(only for } m>0, \left|\vec{v}\right|<1\text{)}, \tag{2.100}$$

the following differences exist:

- Particle charge $\widetilde{q}$, target charge $\widetilde{Q}$, and invariant mass m_0 may be rotated against one another in $\mathbb{C}_0$ by arbitrary real angles. In the classical case, charges may only have an opposite sign. For the purpose of this paper this generalization to $\mathbb{C}_0$ has no predictive value, since only the real absolute values of products $\widetilde{q}\widetilde{Q}$ or $\widetilde{q}\widetilde{Q}m_0$ appear in the $d\sigma/d\Omega$ result, making it impossible to separate the individual factors into their respective magnitudes and phases in $\mathbb{C}_0$.
- The phase ϕ^2 appears in the spin-$\frac{1}{2}$ contribution part $\phi^2\left|\vec{p}\right|^2\sin^2\frac{\theta}{2}$. This makes for a different value in the predicted measurement outcome.

The ratio $r_{\alpha/\text{QED}}$ between general (any α) and classical ($\alpha = \pi/2$) cross section for fixed charges q and Q is:

$$r_{\alpha/\text{QED}} := \frac{d\sigma}{d\Omega}\Big/\left.\frac{d\sigma}{d\Omega}\right|_{\text{QED}} = \frac{\left\|E^2+\phi^2\left|\vec{p}\right|^2\sin^2\frac{\theta}{2}\right\|}{E^2-\left|\vec{p}\right|^2\sin^2\frac{\theta}{2}} = \frac{\sqrt{E^4+2\cos\left(2\alpha\right)E^2\left|\vec{p}\right|^2\sin^2\frac{\theta}{2}+\left|\vec{p}\right|^4\sin^4\frac{\theta}{2}}}{E^2-\left|\vec{p}\right|^2\sin^2\frac{\theta}{2}}. \tag{2.101}$$

On first look, $r_{\alpha/\text{QED}}$ is larger the higher the particle's incident speed is ($E^2 \approx \left|\vec{p}\right|^2$, $\left|\vec{v}\right|^2 \to 1$) and the closer the scattering angle is backwards ($\theta \to \pi$).

2.9. **Approximation for slow moving incident particle.** For slow moving particles, $E \gg \left|\vec{p}\right|$, there is $\left|\vec{v}\right| = \left|\vec{p}\right|/E \ll 1$ and therefore

$$r_{\alpha/\text{QED}}\Big|_{\left|\vec{v}\right|\ll 1} \approx 1 + \left[2\left|\vec{v}\right|^2\cos^2\alpha\right]\sin^2\frac{\theta}{2} + \ldots \tag{2.102}$$

At nonclassical α there is $\cos^2\alpha > 0$ and therefore backscattering ($\theta \approx \pi$) is enhanced by a term proportional to $\left|\vec{v}\right|^2$ and $\cos^2\alpha$. This behavior can be compared, at least qualitatively, with scattering of a classical body in Schwarzschild geometry. This has been investigated [6] and confirms the qualitative agreement (section 4 figure 2 shows enhanced nonvanishing backscattering).

2.10. **Approximation for fast incident particle.** The faster the incoming particle gets, $\left|\vec{v}\right|^2 \to 1$, the smaller the difference between $\left|\vec{p}\right|$ and E becomes. The ratio $r_{\alpha/\text{QED}}$ grows approximately as

$$r_{\alpha/\text{QED}}\Big|_{\left|\vec{v}\right|\to 1} \approx \left[\frac{\cos^2\alpha}{1-\left|\vec{v}\right|}\right]\sin^2\frac{\theta}{2}. \tag{2.103}$$

As $\left|\vec{v}\right|$ approaches the speed of light, scattering becomes infinitely much stronger for general α relative to the conventional case where $\cos^2\alpha = 0$, growing asymptotically like $1/x$ (with $x = 1 - \left|\vec{v}\right|$). This might become a window to observability from quantum gravitational effects, as all current assumptions for particle scattering assume a negligible contribution from quantum gravity. The effect can be compared qualitatively with the spacial distribution of hot inter- and intragalactic gas, and estimated for lateral momentum transfer of a high-energy neutrino through matter.

2.11. **Approximation for purely gravitational interaction.** Purely gravitational Rutherford scattering of a spin-$\frac{1}{2}$ particle may be estimated with all phases in $\mathbb{C}_0$ zero (including $\alpha = 0$), target charge to be its mass at rest $\widetilde{Q} := M$, and incident particle's charge its total energy $\widetilde{q} := E$, so that equation (2.98) becomes:

$$\left.\frac{d\sigma}{d\Omega}\right|_{\text{grav}} = \frac{M^2E^2}{4\,|\vec{p}|^4 \sin^4\frac{\theta}{2}}\left(E^2 + |\vec{p}|^2 \sin^2\frac{\theta}{2}\right) = \frac{M^2}{4\,|\vec{v}|^4 \sin^4\frac{\theta}{2}} \cdot \left(1 + |\vec{v}|^2 \sin^2\frac{\theta}{2}\right). \tag{2.104}$$

As expected from the classical case, the purely gravitational elastic scattering trajectory of the particle only depends on target charge M and particle speed $\vec{v}$, but not the particle's gravitational charge[1]. For fast moving particles $|\vec{v}| \approx 1$ this becomes:

$$\left.\frac{d\sigma}{d\Omega}\right|_{\text{grav},|\vec{v}|\approx 1} \approx \frac{M^2}{4\sin^4\frac{\theta}{2}} \cdot \left(1 + \sin^2\frac{\theta}{2}\right) = \frac{M^2}{4}\left(\frac{1}{\sin^4\frac{\theta}{2}} + \frac{1}{\sin^2\frac{\theta}{2}}\right). \tag{2.105}$$

In order to compare this effect with the strength of other forces, equation (2.105) has to be multiplied with $G^2/c^4 \approx 5.5 * 10^{-55}\,\text{m}^2/\text{kg}^2$ to obtain a magnitude in SI units. For (near) pointlike targets such as electrons ($M_{e^-} \approx 9.1 * 10^{-31}\,\text{kg}$) or neutrons ($M_n \approx 1.7 * 10^{-27}\,\text{kg}$) this effect has a characteristic length scale GM/c^2 in the order of $\sim 10^{-53\ldots-56}\,\text{m}$. It will therefore be overshadowed by electromagnetic interaction and inelastic scattering, where possible. Observable effects can only be expected in the absence of these interactions (e.g. elastic scattering of neutrinos on a point mass) or at very large scales (e.g. elastic scattering of particles with gas in and around galaxies).

2.12. **Estimate for momentum transfer of a fast fermion.** This section estimates the momentum transfer during elastic scattering of a fast incoming point-like fermion on a stationary fermion target. Target recoil is neglected for simplification, as it will not change the order of magnitude of the momentum transfer.

The radially symmetric cross section σ_m is the area within which an incident particle will be scattered at a minimum outgoing angle θ_m. Writing $d\Omega = 2\pi \sin\theta\, d\theta$, equation (2.105) can be integrated:

$$\sigma_m = \int_0^{\sigma_m} d\sigma = \int_{\theta_m}^{\pi} \frac{M^2}{4\sin^4\frac{\theta}{2}} \cdot \left(1 + \sin^2\frac{\theta}{2}\right) 2\pi \sin\theta\, d\theta. \tag{2.106}$$

Using

$$\int \frac{\sin\theta}{\sin^4\frac{\theta}{2}} d\theta = -\frac{2}{\sin^2\frac{\theta}{2}} + \text{const}, \tag{2.107}$$

$$\int \frac{\sin\theta}{\sin^2\frac{\theta}{2}} d\theta = 2\int \cot\frac{\theta}{2}\, d\theta = 4\ln\left(\sin\frac{\theta}{2}\right) + \text{const}, \tag{2.108}$$

this becomes:

$$\sigma_m = \frac{\pi M^2}{2}\left[-\frac{2}{\sin^2\frac{\theta}{2}} + 4\ln\left(\sin\frac{\theta}{2}\right)\right]_{\theta_m}^{\pi} \tag{2.109}$$

$$= \frac{\pi M^2}{2}\left[(-2+0) - \left(-\frac{2}{\sin^2\frac{\theta_m}{2}} + 4\ln\left(\sin\frac{\theta_m}{2}\right)\right)\right] \tag{2.110}$$

$$= \pi M^2\left[\frac{1}{\sin^2\frac{\theta_m}{2}} - \left(1 + \ln\left(\sin^2\frac{\theta_m}{2}\right)\right)\right]. \tag{2.111}$$

Writing Δp_θ for momentum transfer at a given angle, and relative momentum transfer $\Delta\hat{p}_\theta$ as

$$\Delta p_\theta = 2\,|\vec{p}| \sin\frac{\theta}{2}, \tag{2.112}$$

$$\Delta\hat{p}_\theta := \frac{\Delta p_\theta}{2\,|\vec{p}|} = \sin\frac{\theta}{2}, \tag{2.113}$$

there is:

$$\sigma_m = \pi M^2\left[\frac{1}{(\Delta\hat{p}_\theta)^2} - \left(1 + \ln(\Delta\hat{p}_\theta)^2\right)\right], \tag{2.114}$$

$$\hat{\sigma}_m := \frac{\sigma_m}{\pi M^2} = \frac{1}{(\Delta\hat{p}_\theta)^2} - \left(1 + \ln(\Delta\hat{p}_\theta)^2\right). \tag{2.115}$$

Here, $\hat{\sigma}_m$ is the cross section relative to πM^2. This can be solved for $\Delta\hat{p}_\theta$ using the Lambert function (product log function) W to:

$$(\Delta\hat{p}_\theta)^2 = \frac{1}{W\left(e^{\hat{\sigma}_m+1}\right)}. \tag{2.116}$$

[1] When particle and target are of comparable energy, e.g. 1 GeV lepton on proton, target recoil would have to be taken into account.

The relative cross section $\hat{\sigma}_m$ is typically very large. For example, assuming σ_m the face area of a neutron with radius $r_n \approx 0.8 * 10^{-15}$m, mass $M_n \approx 1.7 * 10^{-27}$ kg, and $G^2/c^4 \approx 5.5 * 10^{-55}\,\mathrm{m}^2/\mathrm{kg}^2$ there is approximately:

$$(2.117)\qquad \hat{\sigma}_m|_n \approx \frac{\pi r_n^2}{\pi \frac{G^2}{c^4} M_n^2} \approx \frac{\left(0.8 * 10^{-15}\mathrm{m}\right)^2}{5.5 * 10^{-55}\,\mathrm{m}^2 * \left(1.7 * 10^{-27}\right)^2} \approx \frac{0.64 * 10^{-30}}{5.5 * 10^{-55} * 2.9 * 10^{-54}} \approx 4 * 10^{77}.$$

For large argument, the Lambert function is roughly a logarithm, and the relative momentum transfer $\Delta\hat{p}_\theta$ can be estimated:

$$(2.118)\qquad \lim_{x\to\infty} \frac{W(x)}{\ln x} = 1, \qquad \Longrightarrow \Delta\hat{p}_\theta \approx \frac{1}{\sqrt{\ln\left(e^{\hat{\sigma}_m+1}\right)}} \approx \frac{1}{\sqrt{\hat{\sigma}_m}}.$$

The impact parameter b_m is defined to be at the radius of σ_m, i.e. the maximum distance from the direct path through the target:

$$(2.119)\qquad \sigma_m = \pi b_m^2, \qquad \hat{\sigma}_m = \frac{\sigma_m}{\pi M^2} = \frac{b_m^2}{M^2}.$$

This allows to express the approximate relative momentum transfer $\Delta\hat{p}_m$ in terms of a given impact parameter b_m and target mass A as:

$$(2.120)\qquad \Delta\hat{p}_m \approx \frac{M}{b_m}.$$

In SI units, this is:

$$(2.121)\qquad \Delta\hat{p}_m \approx \frac{GM}{c^2 b_m} \approx 7.4 * 10^{-27}\frac{\mathrm{m}}{\mathrm{kg}} * \frac{M}{b_m}.$$

As expected, the value is still very small compared to known particle interactions through the other forces. When the number of interaction partners becomes in the order of 10^{20} or more, or distances are in the order of the diameter of the Milky Way ($\sim 10^{20}$ m) or larger, an effect may become observable. Given the number of uncertainties on the exact workings of such scattering in the real world, the effect here is small enough to not have been noticed yet.

2.13. **Qualitative prediction for intergalactic medium distribution.** Another opportunity for observing effects predicted in this work is over large distance scales. While individual gravitational scattering between particles is very weak, the difference as compared to current model assumptions may become apparent over galactic distance scales. Assuming that the origin of most highly energized particles and gas in the universe is from within galaxies, increased backscattering at higher energies as predicted in the calculations here should - qualitatively - lead to a distribution of intergalactic gas that is hotter and denser near galaxies as compared to theoretical model predictions. Direct measurement is difficult, as such gas typically does not radiate by itself (keywords include e.g. “warm-hot intergalactic medium”, “intracluster medium”, “circumgalactic enrichment”). In recent years, observation of absorption lines in the spectrum of remote quasars (the “Lyman-α forest”) has become a powerful tool to constrain and tune theoretical models of galactic development. It is envisioned that within the next decade or two it should be possible to make quantitative comparisons between these observations and model predictions.

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References

[1] J. Köplinger, Dirac equation on hyperbolic octonions, *Appl. Math. Comput.* **182** (2006), pp. 443-446.
[2] J. Köplinger, Signature of gravity in conic sedenions, *Appl. Math. Comput.* **188** (2007), pp. 942-947.
[3] J. Köplinger, Hypernumbers and relativity, *Appl. Math. Comput.* **188** (2007), pp. 954-969.
[4] J. Köplinger, Gravity and electromagnetism on conic sedenions, *Appl. Math. Comput.* **188** (2007), pp. 948-953.
[5] J. Köplinger, Quantum of area from gravitation on complex octonions, *arXiv*:0812.0212 (2008).
[6] P. A. Collins, R. Delbourgo, R. M. Williams, On the elastic Schwarzschild scattering cross section, *J. Phys. A: Math. Nucl. Gen.* **6** (1973), pp. 161-169

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