# Doubly nilpotent numbers in the 2D plane — a guided tour of PQ space

A research digest of Shuster and Köplinger (2011)


### Abstract

This is a guided tour of the paper **J. A. Shuster and J. Köplinger, "Doubly nilpotent numbers in the 2D plane", *Appl. Math. Comput.* 217 (2011) 7295–7310** (DOI: 10.1016/j.amc.2011.02.021), which introduces *PQ space*: a two-dimensional number system whose two basis elements $p$ and $q$ both have squares that *map to the coordinate origin*, a generalization of the classical nilpotent relation $p^2 = 0$ into a projective map between a multiplicative group with zero and a vector space. The construction yields anticommutator relations $\{p, p\} = \{q, q\} = 0$ reminiscent of algebraic generators of supersymmetry, a four-leaved-clover locus for real powers of the basis elements, four distinguishable "directed zeros" at the origin, and a signature butterfly-shaped Mandelbrot-type fractal. PQ space is the companion construction to W space (published one year earlier by the same authors); both critique-and-reconstruct a Musean hypernumber intuition on rigorous algebraic footing. Follow-on work is referenced in the body. The digest takes the published journal paper as the authoritative reference.



*Digest author*: Jens Köplinger, 2026. *Produced*: 20 April 2026. This digest was generated with the help of AI and has been reviewed and edited by the original authors for accuracy and relevance.




## 1 At a glance

PQ space is a two-dimensional number system with two basis elements $p$ and $q$ whose squares both *map to the coordinate origin*, generalizing nilpotence without insisting on the literal identity $p^2 = 0$. The construction:

- A **vector space** $\mathrm{PQ}^+$ with basis $\{p, q\}$ carries addition.
- A separate **multiplicative group with zero** $\mathrm{PQ}^\times$ carries multiplication, and its elements map into the complex plane.
- The two are linked by a **projective map** whose kernel identifies elements of $\mathrm{PQ}^\times$ with coordinate rays in $\mathrm{PQ}^+$. Under this linkage, $p^2 \mapsto (0, 0)$ and $q^2 \mapsto (0, 0)$: the squares of the basis elements land at the origin — but the origin now carries structure.

The construction yields:

- Anticommutator relations $\{p, p\} = \{q, q\} = 0$, the algebraic fingerprint of fermionic generators such as the ones used in supersymmetry.

- A **four-leaved clover** as the locus of real powers $p^a$ and $q^a$.
- **"Directed zeros"** — four distinguishable non-zero elements of $\mathrm{PQ}^\times$ whose image under the projection all land on a single point $(0,0) \in \mathrm{PQ}^+$.
- A **butterfly-shaped Mandelbrot fractal** generated by PQ-space recursion — the paper's signature visual artifact (Figures 8.1 and 8.2).

PQ space is the companion construction to W space [SK2010], published one year earlier in the same journal by the same authors. Together they form a methodological pair: the same critique-and-reconstruct approach applied to two different Musean hypernumbers. A separate digest covers W space.

## 2 What is "doubly nilpotent"?

In standard usage, a nilpotent element $n$ of an algebra satisfies $n^2 = 0$ (or more generally $n^k = 0$ for some $k$). Classical examples include:

- **Dual numbers**: basis $\{1, \varepsilon\}$ with $\varepsilon^2 = 0$. Used in automatic differentiation and in Grassmann-algebra constructions.
- **Split-quaternions, split-octonions, and related algebras**: contain nilpotent elements as part of a richer structure, with current interest in connection with supersymmetry in physics.

Shuster and Köplinger [SK2011] construct a 2D system with **two** nilpotent-like basis elements — $p$ and $q$ — whose squares both vanish, but in a generalized sense: $p^2$ and $q^2$ both *map to the coordinate origin* of the underlying vector space, rather than strictly equaling zero as algebraic elements. This is what the authors mean by "**doubly nilpotent**": two independent directions of nilpotence, implemented via a projective map rather than a direct algebraic identity. The construction relaxes the classical relation $A^2 = 0$ to the map

$$A^2 \;\mapsto\; (0,0), \tag{2.1}$$

for $A = y_A u$ along any "nilpotent direction" $u$.

## 3 The construction in three moves

### 3.1 The additive side: $\mathrm{PQ}^+$

First, the authors define the **additive vector space** $\mathrm{PQ}^+$ as $\mathbb{R}^2$ with basis $\{p, q\}$ and ordinary vector addition,

$$(x_1 p + y_1 q) + (x_2 p + y_2 q) = (x_1 + x_2)p + (y_1 + y_2)q. \tag{3.1}$$

This is the "coordinate plane" of PQ space. No multiplication is defined here.

## 3.2 The multiplicative side: $\mathrm{PQ}^\times$

Separately, the authors define the **multiplicative group with zero**

$$\mathrm{PQ}^\times = \{[s,t] \ : \ s \geq 0, \ t \in \mathbb{R}\}, \tag{3.2}$$

with multiplication

$$[s,t] \times [s',t'] = [ss', \, t+t']. \tag{3.3}$$

A zero element $[0]$ (any $[0,t]$) is adjoined as absorbing. The modulus $s$ plays a "length-type" role; the angle $t$ plays a cyclic role with period $2\pi$. $\mathrm{PQ}^\times$ is isomorphic to the multiplicative group $\mathbb{C}^\times$ of the complex numbers via a 45° rotation,

$$* : \mathbb{C}^\times \to \mathrm{PQ}^\times, \qquad \sqrt{\|A\|} \to s_A, \ \alpha_A \to t_A - \tfrac{\pi}{4},$$

which identifies

$$\{1, \, i, \, -1, \, -i\} \ \longleftrightarrow \ \{G, \, g, \, -G, \, -g\} \qquad \text{and} \qquad \{\sqrt{i}, \, \sqrt{i^3}, \, \sqrt{i^5}, \, \sqrt{i^7}\} \ \longleftrightarrow \ \{p, \, q, \, -p, \, -q\}.$$

In particular, the multiplicative identity $G = [1,0]$ corresponds to $1 \in \mathbb{C}$, and the basis elements $p = [1, \pi/4]$ and $q = [1, 3\pi/4]$ are the 45°-rotated images of $\sqrt{i}$ and $\sqrt{i^3}$.

## 3.3 Joining them: the projective map

The link between $\mathrm{PQ}^+$ and $\mathrm{PQ}^\times$ is a **projective map**

$$\iota : \mathrm{PQ}^\times \to \mathrm{PQ}^+, \qquad [s,t] \ \mapsto \ \big(x,y\big) = r\big(\cos(t - \tfrac{\pi}{4}), \sin(t - \tfrac{\pi}{4})\big), \tag{3.4}$$

where the Euclidean radius $r$ of the image in $\mathrm{PQ}^+$ is

$$r \ = \ s\,|\sin(2t)|. \tag{3.5}$$

Equation (3.5) is the key formula — it encodes the whole geometry of PQ space. The factor $|\sin(2t)|$ has zeros at $t = 0, \pi/2, \pi, 3\pi/2$, the four cardinal angles of $\mathrm{PQ}^\times$. At those four angles, any $\mathrm{PQ}^\times$ element — regardless of its modulus $s$ — maps to the *origin* of $\mathrm{PQ}^+$. The basis elements $p = [1, \pi/4]$ and $q = [1, 3\pi/4]$ are the images of $\mathrm{PQ}^\times$ points at angles $\pi/4$ and $3\pi/4$; their squares $p^2 = [1, \pi/2]$ and $q^2 = [1, 3\pi/2]$ sit at exactly the cardinal zeros, so

$$p^2 \mapsto (0,0), \qquad q^2 \mapsto (0,0) \quad \text{in } \mathrm{PQ}^+.$$

This is the mechanism that realizes doubly nilpotent behavior without requiring a literal $p^2 = 0$ in a single algebra: the multiplication lives in $\mathrm{PQ}^\times$, the geometry lives in $\mathrm{PQ}^+$, and the projective map between them collapses certain $\mathrm{PQ}^\times$ elements onto the origin of $\mathrm{PQ}^+$.

# 4 The four-leaved clover: locus of real powers

Real powers $p^a$ (and $q^a$) in $\mathrm{PQ}^\times$ map under $\iota$ to a locus in $\mathrm{PQ}^+$ that resembles a **four-leaved clover**. Explicitly, from [SK2011] §3.6,

$$p^a = \Big[1, \, \tfrac{a\pi}{4}; \, |\sin(\tfrac{a\pi}{2})|\Big], \qquad q^a = \Big[1, \, \tfrac{3a\pi}{4}; \, |\sin(\tfrac{3a\pi}{2})|\Big], \qquad a \in \mathbb{R}, \tag{4.1}$$

where the third bracket slot gives the Euclidean radius in $PQ^+$ on the unit power-orbit. The four leaves reflect the four cardinal angles at which the radius function collapses to zero — the same structure that makes $p^2$ and $q^2$ land at the origin. Equivalently, the dimensionless shape function is

$$r_{c2}(\alpha) = |\cos(2\alpha)|, \tag{4.2}$$

with $\alpha$ the angle along the unit orbit in $PQ^+$ coordinates. The clover is a clean visual summary of the projective-map construction; the selection argument for this shape function (as opposed to other candidates such as $\sin\alpha$, $|\sin\alpha|$, or $r_{c3} = \sqrt{2}\sin(2\alpha)\cos\alpha$) is worked out in §§2.2–2.4 and Table 1 of [SK2011], on grounds of continuity, symmetry, and coverage of the full plane.

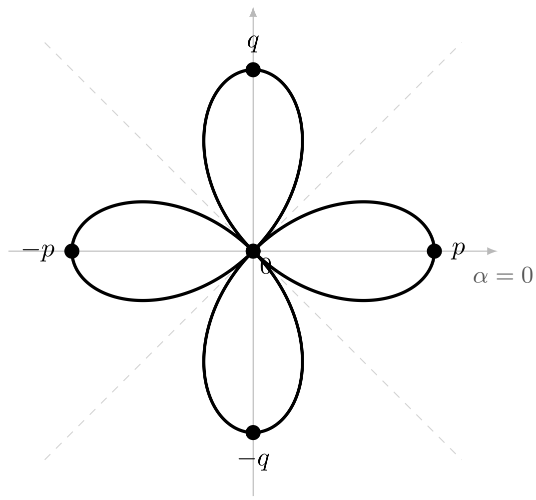


Fig. 4.1: The four-leaved clover in $PQ^+$: image of the unit-modulus orbit of $PQ^\times$ under the projective map $\iota$, with polar radius $r(\alpha) = |\cos(2\alpha)|$. The basis elements $p$ and $q$ sit at the tips of the petals along the coordinate axes; their even integer powers collapse to the origin. The dashed diagonals mark $\alpha = \pm\pi/4, \pm 3\pi/4$, where the clover touches the origin.

Because each of $p$ and $q$ has eight distinct integral powers before returning to the identity,

$$p^1 = p,\ p^2 = g,\ p^3 = q,\ p^4 = -G,\ p^5 = -p,\ p^6 = -g,\ p^7 = -q,\ p^8 = G,$$

the powers cycle with period eight, with every *even* power landing in the zero-center $\Phi$ (see §5).

## 5 The origin is special: directed zeros

In ordinary algebra the origin is a single element — additive identity, and (together with absorption) the multiplicative zero. In PQ space the **origin carries structure**.

The authors introduce a **zero-center**

$$\Phi := \iota^{-1}\big((0,0)\big) \tag{5.1}$$

as the preimage of $(0,0)$ under the projection $\iota$. It splits cleanly into two pieces,

$$\Phi = \{[0,t] \ : \ t \in \mathbb{R}\} \ \cup \ \big\{[s,\, n\pi/2] \ : \ n \in \mathbb{Z},\ s > 0\big\}, \tag{5.2}$$

i.e. the multiplicative zero [0] together with every non-zero $\mathrm{PQ}^{\times}$ element sitting exactly on a cardinal angle. Among the unit-modulus non-zero elements of $\Phi$ are four distinguishable group elements, the **directed zeros**,

$$\{G,\, g,\, -G,\, -g\} \;\subset\; \mathrm{PQ}^{\times}, \tag{5.3}$$

all four of which map to the single point $(0,0) \in \mathrm{PQ}^{+}$. They are non-zero in $\mathrm{PQ}^{\times}$ but indistinguishable in $\mathrm{PQ}^{+}$. Geometrically, the origin of $\mathrm{PQ}^{+}$ is the image of four distinct "zero directions" in $\mathrm{PQ}^{\times}$. Depending on which direction a $\mathrm{PQ}^{\times}$ element approaches the origin from, it retains a distinguishable identity which can re-emerge when multiplied by a non-zero element.

The directed-zeros structure has consequences:

- **Non-distributivity in general.** The joined PQ structure (addition in $\mathrm{PQ}^{+}$ together with the projected multiplication from $\mathrm{PQ}^{\times}$) does not satisfy the full distributive law, because the sum of two directed zeros in $\Phi$ is not in general zero in $\mathrm{PQ}^{\times}$ even though their images in $\mathrm{PQ}^{+}$ are each $(0,0)$. Section 3.7 of [SK2011] works out the precise failure: squaring the formal sum $x_A p + y_A q$ of the two $\mathrm{PQ}^{\times}$ representatives yields a radius-$s_A^2|\sin(4t_A)|$ expression on the left and an identically zero radius on the right, a mismatch except on measure-zero sets.
- **Apparent zero divisors.** Because multiple $\mathrm{PQ}^{\times}$ elements collapse to the origin in $\mathrm{PQ}^{+}$, the joined structure looks as though it has zero divisors. Inside $\mathrm{PQ}^{\times} \setminus [0]$ (a group) it does not — the "divisors" appear only through the projection.

A **diagonal set** $X_0 = \{(x,y) \,:\, x = \pm y,\ y \neq 0\}$ in $\mathrm{PQ}^{+}$ is the preimage-free locus: these points cannot be lifted to $\mathrm{PQ}^{\times}$ under the inverse of (3.5). Products involving $X_0$ elements are therefore left undefined in the coordinate-level multiplication below.

## 6 Coordinate multiplication

The projective construction is algebraically elegant but not computationally transparent. Section 3.9 and Appendix B of [SK2011] derive an explicit **coordinate-level multiplication** on $\mathrm{PQ}^{+}$ that implements the projected $\mathrm{PQ}^{\times}$ multiplication on non-degenerate elements.

For two normalized points $\hat{A} = (x_A, y_A)$ and $\hat{B} = (x_B, y_B)$ with unit modulus, with

$$X := x_A x_B - y_A y_B, \qquad Y := x_A y_B + y_A x_B, \qquad F := \frac{\sqrt{2}\,|XY|}{(X^2+Y^2)^{3/2}},$$

the projected product of scale-$s_A s_B$ form is

$$A \times B \;=\; s_A s_B\, F\,\big(X - Y,\ X + Y\big). \tag{6.1}$$

The $(X,Y)$ vector is the ordinary complex product of $(x_A, y_A)$ and $(x_B, y_B)$; the 45° rotation $(X,Y) \to (X-Y, X+Y)$ implements the isomorphism $* : \mathbb{C}^{\times} \to \mathrm{PQ}^{\times}$; the factor $F$ is the modulus-radius correction from Equation (3.5). The resulting multiplication:

- is **commutative** (inherited from $\mathrm{PQ}^{\times}$ being abelian);
- is **associative in the generic case** (off $\Phi$ and $X_0$);
- is **non-distributive at the origin** (as noted above);
- is **computable numerically** and is used as the kernel of the Mandelbrot-style recursion that generates the butterfly fractal of §8.

## 7 Polynomials, the exponential, and anticommutators

**Polynomials.** Because $p^2$ and $q^2$ both lie in $\Phi$, any polynomial $Z(V) = \sum_n z_n V^n$ in $p, q$ with degree $\geq 2$ in either variable reduces — under the projected multiplication — to a linear combination of $1, p, q, pq$ plus elements of $\Phi$. Section 3.8 of [SK2011] works this out in detail.

**Exponential.** The formal power series

$$\exp V \ := \ \sum_{n=0}^{\infty} \frac{1}{n!} V^n$$

for $V \in \mathrm{PQ}^{\times}$ reduces, after projection through $\iota$, to an expression in which the $|\sin(2nt)|$ factor of each $V^n$ suppresses every even power. The exponential therefore carries only contributions at odd multiples of the base angle, structurally analogous to the exponential in a Grassmann algebra — another setting where higher powers of the "odd" generators collapse.

**Anticommutators.** The algebraic fingerprint of the construction is the pair of relations

$$\{p, p\} = \ 2p^2 \ \mapsto \ (0, 0) \quad \text{in } \mathrm{PQ}^+, \tag{7.1}$$

$$\{q, q\} = \ 2q^2 \ \mapsto \ (0, 0) \quad \text{in } \mathrm{PQ}^+, \tag{7.2}$$

i.e. the anticommutators of each basis element with itself vanish under projection. In physics, this is **the defining relation for a fermionic / anticommuting variable** — the kind of object that appears in the generators of supersymmetry algebras. Section 6 of [SK2011] identifies this as the main physics hook for PQ space, while pointedly noting that the paper does not construct a supersymmetry representation: it only observes that the necessary relations are present, and that the distinction between requiring $p$ to be an *operator* (as in a Clifford algebra) versus a *number* (as here) is offered as a fresh angle for algebraic modeling.

## 8 The butterfly fractal

A signature visual artifact of the paper is the **PQ-space "papillon" (butterfly) fractal**, generated by a Mandelbrot-style recursion using the PQ coordinate multiplication. For a seed point $C = (x_C, y_C) \in \mathrm{PQ}^+$, the iteration is

$$A_0 := C, \qquad A_{n+1} := \left(\left[(A_n)^{-\iota}\right]^2\right)^{\iota} \ + \ C, \tag{8.1}$$

where $(\cdot)^{-\iota}$ lifts from $\mathrm{PQ}^+$ to $\mathrm{PQ}^{\times}$ (where possible), squaring occurs in $\mathrm{PQ}^{\times}$, and $(\cdot)^{\iota}$ projects back to $\mathrm{PQ}^+$. Non-divergent $C$ values are coloured black; the result resembles a butterfly, symmetric about the line $y = -x$ (the axis of the multiplicative identity $G$). The fractal appears as Figure 8 of [SK2011]; full algorithmic documentation is in its Appendix A.

## 9 Relationship to Musès' "$p$ and $q$ numbers"

**Charles Arthur Musès** (1919–2000) was an American philosopher and independent mathematician who, across a sequence of papers in *Applied Mathematics and Computation* and in the *Journal*

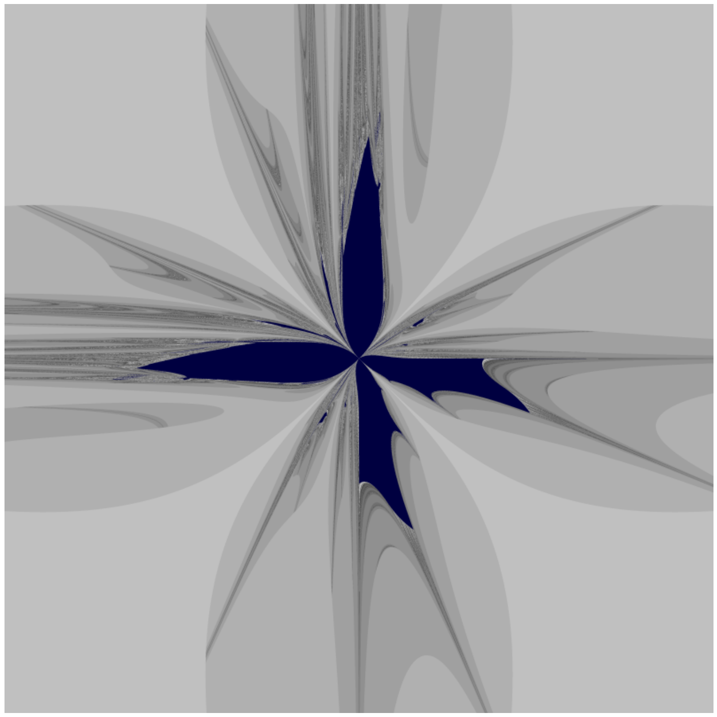

Fig. 8.1: Full “papillon” butterfly fractal, generated by the Mandelbrot-style PQ-space recursion $A_{n+1} = A_n^2 + C$ over the $\{p, q\}$ plane. Non-divergent seed points are coloured black. The symmetry axis $y = -x$ coincides with the multiplicative identity line. Image: J. Köplinger, *PQ Space Mandelbrot* gallery (reused under the author’s CC-BY 4.0 re-licensing for this project from the original CC-BY-SA 3.0 posting on [KoeplWWW]). Source code: [PQSpaceMandelbrot].

*for the Study of Consciousness* between 1968 and 1983, proposed a family of extensions to the complex numbers under the name *hypernumbers* [Muses1968, Muses1972, Muses1978, Muses1979, Muses1983]. His program included “$w$ numbers” (relevant to the companion paper [SK2010]), “$p$ and $q$ numbers” (relevant here), “$m$ numbers”, “epsilon numbers”, and others. Musès earned his PhD from Columbia in 1951 with a dissertation on the 17th-century theosophists Böhme and Freher; he worked outside standard academic physics, edited the *Journal for the Study of Consciousness* (1968–1973), and is also associated with a spiritual practice he called the *Lion Path*. The German Wikipedia entry on Musès [WikiMusesDE] notes explicitly that “Köplinger und Shuster haben seine Ideen aufgegriffen und einige Hyperzahlen genauer analysiert” (“Köplinger and Shuster have taken up his ideas and analyzed several hypernumbers in detail”).

**What Musès proposed.** Across his papers, Musès invoked basis elements $p$ and $q$ satisfying $p^2 = 0$ (with $p \neq 0$), $|p| = 1$, and lying along an axis “perpendicular to any ordinary imaginary unit”. The geometric intuition was two nilpotent directions in a flat 2D plane, with the origin playing a distinguished role. In different papers he proposed different shapes for the unit power-orbit: first

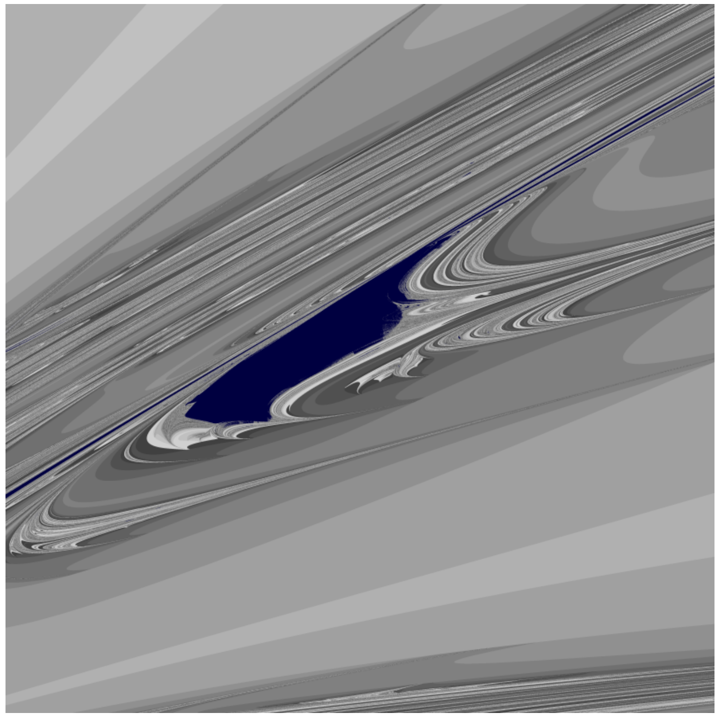

Fig. 8.2: Zoom into a detail region of the butterfly, range $\{[0.25, 0.45], [0.1, 0.3]\}$ with iteration depth 5000. Self-similar filigree along the branch structure is visible at this scale. Image from the author's CC-BY 4.0 gallery [KoeplWWW].

a quadrifolium with Cartesian form $(x^2 + y^2)^3 = (x^2 - y^2)^2$ (i.e. essentially $r = |\cos(2\alpha)|$, the four-leaved clover retained in [SK2011]), and later [Muses1978, Muses1979, Muses1983] a different shape given in polar form by $r = \cos(2\theta)(\cos\theta - \sin\theta)$.

**The Shuster–Köplinger stance.** The authors of [SK2011] credit Musès with the **preceding idea** — two nilpotent directions in a 2D plane, with a structured origin — but document, with explicit citations to Musès' published work, that his formulation was **inconsistent and underspecified**:

- the claimed relations $p^2 = 0$ and "$p^0 = 0$" taken simultaneously produce contradictory conclusions: combining $P_1 = 0$ (from $p^0 = 0$ and $r_1 = 1$) with $P_1 P_2 = r_2 p^{k_2} = P_2$ would force $P_2 = 0$, which is not the envisioned system (cf. [SK2011] §4);
- the role of the origin was invoked informally but never formalized;
- the relation between algebraic (multiplicative) and geometric (coordinate-plane) representations was never stated precisely, so the multiplication was not closed.

PQ space is offered as a **clean, formal replacement** that preserves what was right about Musès' intuition — two nilpotent directions, a structured origin, the four-leaved clover — while giving it a consistent algebraic foundation via the projective-map construction. This is the same methodological pattern as the companion paper [SK2010] on W space: *advocate for Musès as a source of mathematical intuition, critique specific claims that fail under scrutiny, and provide a fully rigorous replacement.*

## 10 Extensions sketched in the paper

Section 5 of [SK2011] lists two extensions left as open directions.

**A dual multiplication PQ$^\circ$.** Parallel to the dual-multiplication construction in W space [SK2010], an alternative multiplicative group PQ$^\circ$ can be defined whose identity element is $-G = [1, \pi; 0]$, with angles measured *clockwise* from the $q$-axis. The resulting multiplication $\circ$ is a mirror image of $\times$ across the line $x = y$, and satisfies $A \times B = -(A \circ B)$ on non-degenerate points. Combining PQ$^\times$ and PQ$^\circ$ in the style of W space's $+\mathbb{W}$ and $-\mathbb{W}$ is noted as a natural next direction.

**Addition in $\Phi$.** A separate additive structure can be imposed on the zero-center $\Phi$ itself, giving the four directed zeros a small-scale group structure of their own, and thereby replacing the non-invertible projection (3.5) by a generalized map with trivial kernel.

Neither extension is fully developed in [SK2011]. Both are noted as natural next steps for future work in the PQ-space style.

## 11 Reception in the literature

As of a consolidated cited-by audit of the literature, **no substantive mathematical follow-up to PQ space has appeared in the peer-reviewed literature**. The paper appears in citation lists of review-style articles in hyperbolic-octonion, sedenion, and related field theory (Demir, Tanışlı, Kansu and collaborators at Anadolu University; Mironov's "sedeons"; Weng; Chanyal; Panicaud) and in pure-algebra studies of split/dual-quaternion variants, always as part of a broader reference set on "non-standard 2D number systems" rather than as a structural input to new mathematics.

This modest reception should be weighed against the paper's scope: the construction is algebraically self-contained and visually distinctive, but the physics hook (supersymmetry) is an *observation* rather than a calculation.

## 12 How to read the original

The paper was published in *Applied Mathematics and Computation* **217** (2011) 7295–7310, DOI: 10.1016/j.amc.2011.02.021. Personal versions of the authors' work are available on their personal pages and on ResearchGate; see [KoeplWWW, ResearchGateJK, ResearchGateJS].

The paper is organized as follows:

| Section | Content |
|---|---|
| §1 Introduction | Nilpotent 2D systems; supersymmetry hook |
| §2 Generalizations | Dual numbers; power-orbit constraints; Table 1 of candidate radius functions; selection of $r_{c2} = \lvert\cos(2\alpha)\rvert$ |
| §3 PQ space | $\mathrm{PQ}^+$, $\mathrm{PQ}^\times$, projection $\iota$, zero-center, directed zeros, coordinate multiplication |
| §4 Musès $p$, $q$ numbers | Comparison with Musès' earlier proposals |
| §5 Extensions | Dual multiplication $\mathrm{PQ}^\circ$; addition in $\Phi$ |
| §6 Summary | Applications hinted: supersymmetry, chaos/catastrophe |
| Appx. A | Butterfly Mandelbrot fractal (Figure 8) |
| Appx. B | Derivation of the coordinate multiplication |

## 13 About the authors

**John A. Shuster** is an independent researcher. His publications are available on ResearchGate [ResearchGateJS].

**Jens Köplinger** is the author of this digest. His personal page with publications, preprints, fractal galleries, and audio renderings is at [KoeplWWW]; his ResearchGate profile is [ResearchGateJK].

## 14 How to cite

For the original research, cite the published journal paper:

> J. A. Shuster and J. Köplinger, "Doubly nilpotent numbers in the 2D plane", *Appl. Math. Comput.* **217** (2011) 7295–7310. DOI: 10.1016/j.amc.2011.02.021.

For this digest, please refer to the ResearchGate entry on the author's profile [ResearchGateJK].

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