

# Doubly nilpotent numbers in the 2D plane — a guided tour of PQ space

A research digest of Shuster and Köplinger (2011)

## Abstract

This is a guided tour of the paper **J. A. Shuster and J. Köplinger**, “Doubly nilpotent numbers in the 2D plane”, *Appl. Math. Comput.* **217** (2011) **7295–7310** (DOI: 10.1016/j.amc.2011.02.021), which introduces *PQ space*: a two-dimensional number system whose two basis elements  $p$  and  $q$  both have squares that *map to the coordinate origin*, a generalization of the classical nilpotent relation  $p^2 = 0$  into a projective map between a multiplicative group with zero and a vector space. The construction yields anticommutator relations  $\{p, p\} = \{q, q\} = 0$  reminiscent of algebraic generators of supersymmetry, a four-leaved-clover locus for real powers of the basis elements, four distinguishable “directed zeros” at the origin, and a signature butterfly-shaped Mandelbrot-type fractal. PQ space is the companion construction to W space (published one year earlier by the same authors); both critique-and-reconstruct a Musean hypernumber intuition on rigorous algebraic footing. Follow-on work is referenced in the body. The digest takes the published journal paper as the authoritative reference.

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## 1 At a glance

PQ space is a two-dimensional number system with two basis elements  $p$  and  $q$  whose squares both *map to the coordinate origin*, generalizing nilpotence without insisting on the literal identity  $p^2 = 0$ . The construction:

- A **vector space**  $PQ^+$  with basis  $\{p, q\}$  carries addition.
- A separate **multiplicative group with zero**  $PQ^\times$  carries multiplication, and its elements map into the complex plane.
- The two are linked by a **projective map** whose kernel identifies elements of  $PQ^\times$  with coordinate rays in  $PQ^+$ . Under this linkage,  $p^2 \mapsto (0, 0)$  and  $q^2 \mapsto (0, 0)$ : the squares of the basis elements land at the origin — but the origin now carries structure.

The construction yields:

- Anticommutator relations  $\{p, p\} = \{q, q\} = 0$ , the algebraic fingerprint of fermionic generators such as the ones used in supersymmetry.

- A **four-leaved clover** as the locus of real powers  $p^a$  and  $q^a$ .
- “**Directed zeros**” — four distinguishable non-zero elements of  $PQ^\times$  whose image under the projection all land on a single point  $(0, 0) \in PQ^+$ .
- A **butterfly-shaped Mandelbrot fractal** generated by PQ-space recursion — the paper’s signature visual artifact (Figures 8.1 and 8.2).

PQ space is the companion construction to W space [SK2010], published one year earlier in the same journal by the same authors. Together they form a methodological pair: the same critique-and-reconstruct approach applied to two different Musean hypernumbers. A separate digest covers W space.

## 2 What is “doubly nilpotent”?

In standard usage, a nilpotent element  $n$  of an algebra satisfies  $n^2 = 0$  (or more generally  $n^k = 0$  for some  $k$ ). Classical examples include:

- **Dual numbers**: basis  $\{1, \varepsilon\}$  with  $\varepsilon^2 = 0$ . Used in automatic differentiation and in Grassmann-algebra constructions.
- **Split-quaternions, split-octonions, and related algebras**: contain nilpotent elements as part of a richer structure, with current interest in connection with supersymmetry in physics.

Shuster and Köpflinger [SK2011] construct a 2D system with **two** nilpotent-like basis elements —  $p$  and  $q$  — whose squares both vanish, but in a generalized sense:  $p^2$  and  $q^2$  both *map to the coordinate origin* of the underlying vector space, rather than strictly equaling zero as algebraic elements. This is what the authors mean by “**doubly nilpotent**”: two independent directions of nilpotence, implemented via a projective map rather than a direct algebraic identity. The construction relaxes the classical relation  $A^2 = 0$  to the map

$$A^2 \mapsto (0, 0), \tag{2.1}$$

for  $A = y_A u$  along any “nilpotent direction”  $u$ .

## 3 The construction in three moves

It may help to keep the following parallel in mind throughout. The complex numbers  $\mathbb{C}$  are simultaneously a vector space  $\mathbb{C}^+$  and a multiplicative group with zero  $\mathbb{C}^\times$ , identified trivially: a point’s Euclidean radius and its multiplicative modulus are the same number. The construction below takes the same two halves but builds them **independently** as  $PQ^+$  (the additive analog of  $\mathbb{C}^+$ ) and  $PQ^\times$  (the multiplicative analog of  $\mathbb{C}^\times$ ), and then re-joins them via a **non-trivial** projection  $\iota$  in place of the trivial radius-equals-modulus identification of  $\mathbb{C}$ . The three moves below implement this split-and-rejoin literally.

### 3.1 The additive side: $PQ^+$ (analog of $\mathbb{C}^+$ )

First, the authors define the **additive vector space**  $PQ^+$  as  $\mathbb{R}^2$  with basis  $\{p, q\}$  and ordinary vector addition,

$$(x_1p + y_1q) + (x_2p + y_2q) = (x_1 + x_2)p + (y_1 + y_2)q. \quad (3.1)$$

This is the “coordinate plane” of  $PQ$  space — the analog of  $\mathbb{C}$  regarded purely as a 2D vector space  $\mathbb{C}^+$ , with no multiplication attached.

### 3.2 The multiplicative side: $PQ^\times$ (analog of $\mathbb{C}^\times$ )

Separately, the authors define the **multiplicative group with zero**

$$PQ^\times = \{[s, t] : s \geq 0, t \in \mathbb{R}\} \quad (3.2)$$

as the analog of  $\mathbb{C}^\times$ , with multiplication

$$[s, t] \times [s', t'] = [ss', t + t']. \quad (3.3)$$

A zero element  $[0]$  (any  $[0, t]$ ) is adjoined as absorbing. The modulus  $s$  plays a “length-type” role; the angle  $t$  plays a cyclic role with period  $2\pi$ .  $PQ^\times$  is isomorphic to  $\mathbb{C}^\times$  as a group via a  $45^\circ$  rotation,

$$* : \mathbb{C}^\times \rightarrow PQ^\times, \quad \sqrt{\|A\|} \rightarrow s_A, \quad \alpha_A \rightarrow t_A - \frac{\pi}{4},$$

which identifies

$$\{1, i, -1, -i\} \longleftrightarrow \{G, g, -G, -g\} \quad \text{and} \quad \{\sqrt{i}, \sqrt{i^3}, \sqrt{i^5}, \sqrt{i^7}\} \longleftrightarrow \{p, q, -p, -q\}.$$

In particular, the multiplicative identity  $G = [1, 0]$  corresponds to  $1 \in \mathbb{C}$ , and the basis elements  $p = [1, \pi/4]$  and  $q = [1, 3\pi/4]$  are the  $45^\circ$ -rotated images of  $\sqrt{i}$  and  $\sqrt{i^3}$ .

### 3.3 Joining them: the projective map

The link between  $PQ^+$  and  $PQ^\times$  is a **projective map**

$$\iota : PQ^\times \rightarrow PQ^+, \quad [s, t] \mapsto (x, y) = r(\cos(t - \frac{\pi}{4}), \sin(t - \frac{\pi}{4})), \quad (3.4)$$

where the Euclidean radius  $r$  of the image in  $PQ^+$  is

$$r = s |\sin(2t)|. \quad (3.5)$$

This is the move that replaces the trivial radius-equals-modulus identification of  $\mathbb{C}$  with something genuinely non-trivial: the multiplicative modulus  $s$  in  $PQ^\times$  and the Euclidean radius  $r$  in  $PQ^+$  are no longer the same number for a generic element. Equation (3.5) is the key formula — it encodes the whole geometry of  $PQ$  space. The factor  $|\sin(2t)|$  has zeros at  $t = 0, \pi/2, \pi, 3\pi/2$ , the four cardinal angles of  $PQ^\times$ . At those four angles, any  $PQ^\times$  element — regardless of its modulus  $s$  — maps to the *origin* of  $PQ^+$ . The basis elements  $p = [1, \pi/4]$  and  $q = [1, 3\pi/4]$  are the images of  $PQ^\times$  points at angles  $\pi/4$  and  $3\pi/4$ ; their squares  $p^2 = [1, \pi/2]$  and  $q^2 = [1, 3\pi/2]$  sit at exactly the cardinal zeros, so

$$p^2 \mapsto (0, 0), \quad q^2 \mapsto (0, 0) \quad \text{in } PQ^+.$$

This is the mechanism that realizes doubly nilpotent behavior without requiring a literal  $p^2 = 0$  in a single algebra: the multiplication lives in  $PQ^\times$ , the geometry lives in  $PQ^+$ , and the projective map between them collapses certain  $PQ^\times$  elements onto the origin of  $PQ^+$ . Figure 3.1 traces a representative point through the two maps.

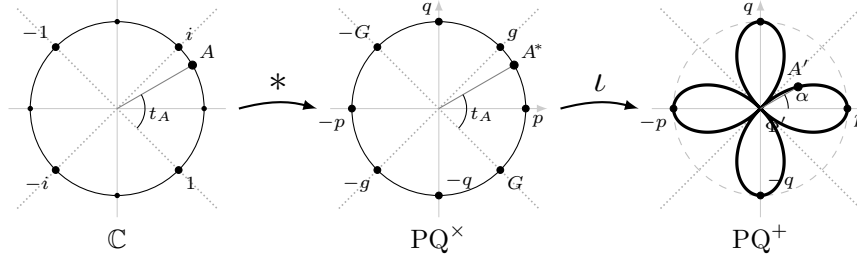


Fig. 3.1: The maps  $*$  :  $\mathbb{C}^\times \rightarrow \text{PQ}^\times$  and  $\iota$  :  $\text{PQ}^\times \rightarrow \text{PQ}^+$ . **Left:** a point  $A = [s_A, t_A]$  on the unit circle of  $\mathbb{C}$ , with the multiplicative identity 1 drawn at panel angle  $-\pi/4$  so that the arc from 1 to  $A$  measures the parameter  $t_A$ . **Middle:** its image  $A^* = [s_A, t_A]$  under the multiplicative-group isomorphism, on the unit-modulus circle of  $\text{PQ}^\times$ , with the multiplicative identity  $G$  at the same panel angle; the four cardinal-angle elements  $\{G, g, -G, -g\}$  are the directed zeros. **Right:** its image  $A' = (x_A, y_A)$  under the projection  $\iota$ , on the four-leaved clover of  $\text{PQ}^+$ , at panel angle  $\alpha = t_A - \pi/4$  and Euclidean radius  $r = s_A |\sin(2t_A)|$ . All four directed zeros of  $\text{PQ}^\times$  collapse onto the single zero-center  $\Phi' = (0, 0)$  of  $\text{PQ}^+$ . Redrawn after Figure 7 of [SK2011].

#### 4 The four-leaved clover: locus of real powers

Real powers  $p^a$  (and  $q^a$ ) in  $\text{PQ}^\times$  map under  $\iota$  to a locus in  $\text{PQ}^+$  that resembles a **four-leaved clover**. Explicitly, from [SK2011] §3.6,

$$p^a = \left[ 1, \frac{a\pi}{4}; |\sin(\frac{a\pi}{2})| \right], \quad q^a = \left[ 1, \frac{3a\pi}{4}; |\sin(\frac{3a\pi}{2})| \right], \quad a \in \mathbb{R}, \quad (4.1)$$

where the third bracket slot gives the Euclidean radius in  $\text{PQ}^+$  on the unit power-orbit. The four leaves reflect the four cardinal angles at which the radius function collapses to zero — the same structure that makes  $p^2$  and  $q^2$  land at the origin. Equivalently, the dimensionless shape function is

$$r_{c2}(\alpha) = |\cos(2\alpha)|, \quad (4.2)$$

with  $\alpha$  the angle along the unit orbit in  $\text{PQ}^+$  coordinates. The clover is a clean visual summary of the projective-map construction; the selection argument for this shape function (as opposed to other candidates such as  $\sin \alpha$ ,  $|\sin \alpha|$ , or  $r_{c3} = \sqrt{2} \sin(2\alpha) \cos \alpha$ ) is worked out in §§2.2–2.4 and Table 1 of [SK2011], on grounds of continuity, symmetry, and coverage of the full plane.

Because each of  $p$  and  $q$  has eight distinct integral powers before returning to the identity,

$$p^1 = p, p^2 = g, p^3 = q, p^4 = -G, p^5 = -p, p^6 = -g, p^7 = -q, p^8 = G,$$

the powers cycle with period eight, with every *even* power landing in the zero-center  $\Phi$  (see §5).

#### 5 The origin is special: directed zeros

In ordinary algebra the origin is a single element — additive identity, and (together with absorption) the multiplicative zero. In PQ space the **origin carries structure**.

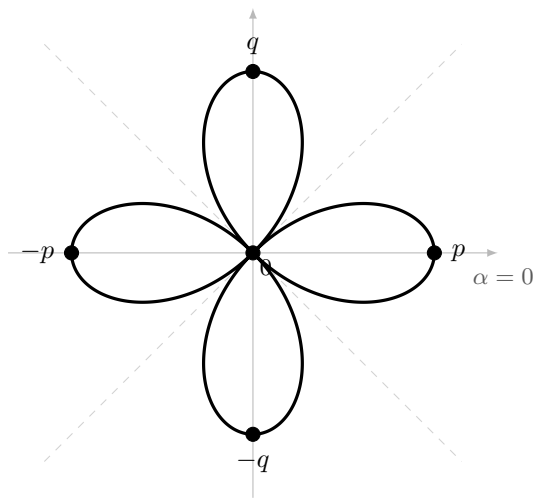


Fig. 4.1: The four-leaved clover in  $PQ^+$ : image of the unit-modulus orbit of  $PQ^\times$  under the projective map  $\iota$ , with polar radius  $r(\alpha) = |\cos(2\alpha)|$ . The basis elements  $p$  and  $q$  sit at the tips of the petals along the coordinate axes; their even integer powers collapse to the origin. The dashed diagonals mark  $\alpha = \pm\pi/4, \pm3\pi/4$ , where the clover touches the origin.

The authors introduce a **zero-center**

$$\Phi := \iota^{-1}((0, 0)) \quad (5.1)$$

as the preimage of  $(0, 0)$  under the projection  $\iota$ . It splits cleanly into two pieces,

$$\Phi = \{[0, t] : t \in \mathbb{R}\} \cup \{[s, n\pi/2] : n \in \mathbb{Z}, s > 0\}, \quad (5.2)$$

i.e. the multiplicative zero  $[0]$  together with every non-zero  $PQ^\times$  element sitting exactly on a cardinal angle. Among the unit-modulus non-zero elements of  $\Phi$  are four distinguishable group elements, the **directed zeros**,

$$\{G, g, -G, -g\} \subset PQ^\times, \quad (5.3)$$

all four of which map to the single point  $(0, 0) \in PQ^+$ . They are non-zero in  $PQ^\times$  but indistinguishable in  $PQ^+$ . Geometrically, the origin of  $PQ^+$  is the image of four distinct “zero directions” in  $PQ^\times$ . Depending on which direction a  $PQ^\times$  element approaches the origin from, it retains a distinguishable identity which can re-emerge when multiplied by a non-zero element.

The directed-zeros structure has consequences:

- **Non-distributivity in general.** The joined  $PQ$  structure (addition in  $PQ^+$  together with the projected multiplication from  $PQ^\times$ ) does not satisfy the full distributive law, because the sum of two directed zeros in  $\Phi$  is not in general zero in  $PQ^\times$  even though their images in  $PQ^+$  are each  $(0, 0)$ . Section 3.7 of [SK2011] works out the precise failure: squaring the formal sum  $x_A p + y_A q$  of the two  $PQ^\times$  representatives yields a radius- $s_A^2 |\sin(4t_A)|$  expression on the left and an identically zero radius on the right, a mismatch except on measure-zero sets.

- **Apparent zero divisors.** Because multiple  $\text{PQ}^\times$  elements collapse to the origin in  $\text{PQ}^+$ , the joined structure looks as though it has zero divisors. Inside  $\text{PQ}^\times \setminus [0]$  (a group) it does not — the “divisors” appear only through the projection.

A **diagonal set**  $X_0 = \{(x, y) : x = \pm y, y \neq 0\}$  in  $\text{PQ}^+$  is the preimage-free locus: these points cannot be lifted to  $\text{PQ}^\times$  under the inverse of (3.5). Products involving  $X_0$  elements are therefore left undefined in the coordinate-level multiplication below.

## 6 Coordinate multiplication

The projective construction is algebraically elegant but not computationally transparent. Section 3.9 and Appendix B of [SK2011] derive an explicit **coordinate-level multiplication** on  $\text{PQ}^+$  that implements the projected  $\text{PQ}^\times$  multiplication on non-degenerate elements.

For two normalized points  $\hat{A} = (x_A, y_A)$  and  $\hat{B} = (x_B, y_B)$  with unit modulus, with

$$X := x_A x_B - y_A y_B, \quad Y := x_A y_B + y_A x_B, \quad F := \frac{\sqrt{2}|XY|}{(X^2 + Y^2)^{3/2}},$$

the projected product of scale- $s_A s_B$  form is

$$A \times B = s_A s_B F (X - Y, X + Y). \quad (6.1)$$

The  $(X, Y)$  vector is the ordinary complex product of  $(x_A, y_A)$  and  $(x_B, y_B)$ ; the  $45^\circ$  rotation  $(X, Y) \rightarrow (X - Y, X + Y)$  implements the isomorphism  $*$  :  $\mathbb{C}^\times \rightarrow \text{PQ}^\times$ ; the factor  $F$  is the modulus-radius correction from Equation (3.5). The resulting multiplication:

- is **commutative** (inherited from  $\text{PQ}^\times$  being abelian);
- is **associative in the generic case** (off  $\Phi$  and  $X_0$ );
- is **non-distributive at the origin** (as noted above);
- is **computable numerically** and is used as the kernel of the Mandelbrot-style recursion that generates the butterfly fractal of §8.

## 7 Polynomials, the exponential, and anticommutators

**Polynomials.** Because  $p^2$  and  $q^2$  both lie in  $\Phi$ , any polynomial  $Z(V) = \sum_n z_n V^n$  in  $p, q$  with degree  $\geq 2$  in either variable reduces — under the projected multiplication — to a linear combination of  $1, p, q, pq$  plus elements of  $\Phi$ . Section 3.8 of [SK2011] works this out in detail.

**Exponential.** The formal power series

$$\exp V := \sum_{n=0}^{\infty} \frac{1}{n!} V^n$$

for  $V \in \text{PQ}^\times$  reduces, after projection through  $\iota$ , to an expression in which the  $|\sin(2nt)|$  factor of each  $V^n$  suppresses every even power. The exponential therefore carries only contributions at odd multiples of the base angle, structurally analogous to the exponential in a Grassmann algebra — another setting where higher powers of the “odd” generators collapse.

**Anticommutators.** The algebraic fingerprint of the construction is the pair of relations

$$\{p, p\} = 2p^2 \mapsto (0, 0) \text{ in } \text{PQ}^+, \quad (7.1)$$

$$\{q, q\} = 2q^2 \mapsto (0, 0) \text{ in } \text{PQ}^+, \quad (7.2)$$

i.e. the anticommutators of each basis element with itself vanish under projection. In physics, this is **the defining relation for a fermionic / anticommuting variable** — the kind of object that appears in the generators of supersymmetry algebras. Section 6 of [SK2011] identifies this as the main physics hook for PQ space, while pointedly noting that the paper does not construct a supersymmetry representation: it only observes that the necessary relations are present, and that the distinction between requiring  $p$  to be an *operator* (as in a Clifford algebra) versus a *number* (as here) is offered as a fresh angle for algebraic modeling.

## 8 The butterfly fractal

A signature visual artifact of the paper is the **PQ-space “papillon” (butterfly) fractal**, generated by a Mandelbrot-style recursion using the PQ coordinate multiplication. For a seed point  $C = (x_C, y_C) \in \text{PQ}^+$ , the iteration is

$$A_0 := C, \quad A_{n+1} := \left( [(A_n)^{-\iota}]^2 \right)^\iota + C, \quad (8.1)$$

where  $(\cdot)^{-\iota}$  lifts from  $\text{PQ}^+$  to  $\text{PQ}^\times$  (where possible), squaring occurs in  $\text{PQ}^\times$ , and  $(\cdot)^\iota$  projects back to  $\text{PQ}^+$ . Non-divergent  $C$  values are coloured black; the result resembles a butterfly, symmetric about the line  $y = -x$  (the axis of the multiplicative identity  $G$ ). The fractal appears as Figure 8 of [SK2011]; full algorithmic documentation is in its Appendix A.

## 9 Relationship to Musès’ “ $p$ and $q$ numbers”

**Charles Arthur Musès** (1919–2000) was an American philosopher and independent mathematician who, across a sequence of papers in *Applied Mathematics and Computation* and in the *Journal for the Study of Consciousness* between 1968 and 1983, proposed a family of extensions to the complex numbers under the name *hypernumbers* [Muses1968, Muses1972, Muses1978, Muses1979, Muses1983]. His program included “ $w$  numbers” (relevant to the companion paper [SK2010]), “ $p$  and  $q$  numbers” (relevant here), “ $m$  numbers”, “epsilon numbers”, and others. Musès earned his PhD from Columbia in 1951 with a dissertation on the 17th-century theosophists Böhme and Freher; he worked outside standard academic physics, edited the *Journal for the Study of Consciousness* (1968–1973), and is also associated with a spiritual practice he called the *Lion Path*. The German Wikipedia entry on Musès [WikiMusesDE] notes explicitly that “Köplinger und Shuster haben seine Ideen aufgegriffen und einige Hyperzahlen genauer analysiert” (“Köplinger and Shuster have taken up his ideas and analyzed several hypernumbers in detail”).

**What Musès proposed.** Across his papers, Musès invoked basis elements  $p$  and  $q$  satisfying  $p^2 = 0$  (with  $p \neq 0$ ),  $|p| = 1$ , and lying along an axis “perpendicular to any ordinary imaginary unit”. The geometric intuition was two nilpotent directions in a flat 2D plane, with the origin playing a distinguished role. In different papers he proposed different shapes for the unit power-orbit: first a quadrifolium with Cartesian form  $(x^2 + y^2)^3 = (x^2 - y^2)^2$  (i.e. essentially  $r = |\cos(2\alpha)|$ ), the

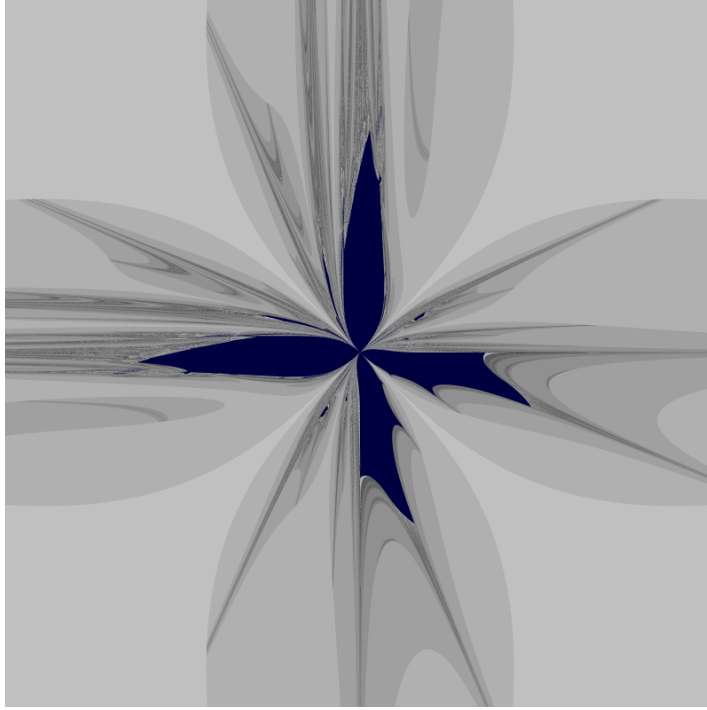


Fig. 8.1: Full “papillon” butterfly fractal, generated by the Mandelbrot-style PQ-space recursion  $A_{n+1} = A_n^2 + C$  over the  $\{p, q\}$  plane. Non-divergent seed points are coloured black. The symmetry axis  $y = -x$  coincides with the multiplicative identity line. Image: J. Köpflinger, *PQ Space Mandelbrot* gallery (reused under the author’s CC-BY 4.0 re-licensing for this project from the original CC-BY-SA 3.0 posting on [KoeplWWW]). Source code: [PQSpaceMandelbrot].

four-leaved clover retained in [SK2011]), and later [Muses1978, Muses1979, Muses1983] a different shape given in polar form by  $r = \cos(2\theta)(\cos \theta - \sin \theta)$ .

**The Shuster–Köpflinger stance.** The authors of [SK2011] credit Musès with the **preceding idea** — two nilpotent directions in a 2D plane, with a structured origin — but document, with explicit citations to Musès’ published work, that his formulation was **inconsistent and underspecified**:

- the claimed relations  $p^2 = 0$  and “ $p^0 = 0$ ” taken simultaneously produce contradictory conclusions: combining  $P_1 = 0$  (from  $p^0 = 0$  and  $r_1 = 1$ ) with  $P_1 P_2 = r_2 p^{k_2} = P_2$  would force  $P_2 = 0$ , which is not the envisioned system (cf. [SK2011] §4);
- the role of the origin was invoked informally but never formalized;
- the relation between algebraic (multiplicative) and geometric (coordinate-plane) representations was never stated precisely, so the multiplication was not closed.

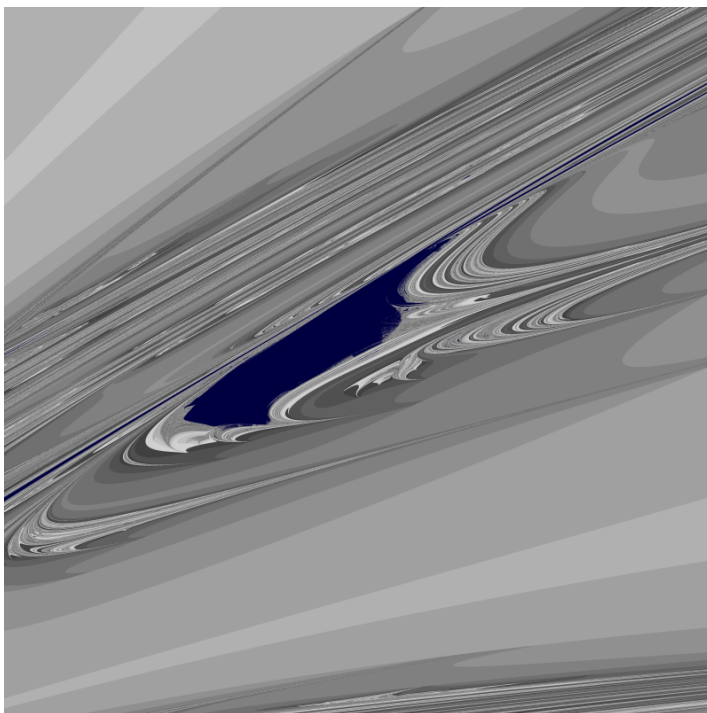


Fig. 8.2: Zoom into a detail region of the butterfly, range  $\{[0.25, 0.45], [0.1, 0.3]\}$  with iteration depth 5000. Self-similar filigree along the branch structure is visible at this scale. Image from the author's CC-BY 4.0 gallery [KoeplWWW].

PQ space is offered as a **clean, formal replacement** that preserves what was right about Musès' intuition — two nilpotent directions, a structured origin, the four-leaved clover — while giving it a consistent algebraic foundation via the projective-map construction. This is the same methodological pattern as the companion paper [SK2010] on W space: *advocate for Musès as a source of mathematical intuition, critique specific claims that fail under scrutiny, and provide a fully rigorous replacement.*

## 10 Extensions sketched in the paper

Section 5 of [SK2011] lists two extensions left as open directions.

**A dual multiplication  $PQ^\circ$ .** Parallel to the dual-multiplication construction in W space [SK2010], an alternative multiplicative group  $PQ^\circ$  can be defined whose identity element is  $-G = [1, \pi; 0]$ , with angles measured *clockwise* from the  $q$ -axis. The resulting multiplication  $\circ$  is a mirror image of  $\times$  across the line  $x = y$ , and satisfies  $A \times B = -(A \circ B)$  on non-degenerate points. Combining  $PQ^\times$  and  $PQ^\circ$  in the style of W space's  $+\mathbb{W}$  and  $-\mathbb{W}$  is noted as a natural next direction.

**Addition in  $\Phi$ .** A separate additive structure can be imposed on the zero-center  $\Phi$  itself, giving the four directed zeros a small-scale group structure of their own, and thereby replacing the non-invertible projection (3.5) by a generalized map with trivial kernel.

Neither extension is fully developed in [SK2011]. Both are noted as natural next steps for future work in the PQ-space style.

## 11 Reception in the literature

As of a consolidated cited-by audit of the literature, **no substantive mathematical follow-up to PQ space has appeared in the peer-reviewed literature**. The paper appears in citation lists of review-style articles in hyperbolic-octonion, sedenion, and related field theory (Demir, Taşlı, Kansu and collaborators at Anadolu University; Mironov’s “sedeons”; Weng; Chanyal; Panicaud) and in pure-algebra studies of split/dual-quaternion variants, always as part of a broader reference set on “non-standard 2D number systems” rather than as a structural input to new mathematics.

This modest reception should be weighed against the paper’s scope: the construction is algebraically self-contained and visually distinctive, but the physics hook (supersymmetry) is an *observation* rather than a calculation.

## 12 How to read the original

The paper was published in *Applied Mathematics and Computation* **217** (2011) 7295–7310, DOI: 10.1016/j.amc.2011.0. Personal versions of the authors’ work are available on their personal pages and on ResearchGate; see [KoeplWWW, ResearchGateJK, ResearchGateJS].

The paper is organized as follows:

Section	Content
§1 Introduction	Nilpotent 2D systems; supersymmetry hook
§2 Generalizations	Dual numbers; power-orbit constraints; Table 1 of candidate radius functions; selection of $r_{c2} =  \cos(2\alpha) $
§3 PQ space	$PQ^+$ , $PQ^\times$ , projection $\iota$ , zero-center, directed zeros, coordinate multiplication
§4 Musès $p$ , $q$ numbers	Comparison with Musès’ earlier proposals
§5 Extensions	Dual multiplication $PQ^\circ$ ; addition in $\Phi$
§6 Summary	Applications hinted: supersymmetry, chaos/catastrophe
Appx. A	Butterfly Mandelbrot fractal (Figure 8)
Appx. B	Derivation of the coordinate multiplication

## 13 About the authors

**John A. Shuster** is an independent researcher. His publications are available on ResearchGate [ResearchGateJS].

**Jens Köpflinger** is the author of this digest. His personal page with publications, preprints, fractal galleries, and audio renderings is at [KoeplWWW]; his ResearchGate profile is [ResearchGateJK].

## 14 How to cite

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For this digest, please refer to the ResearchGate entry on the author’s profile [ResearchGateJK].

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