Elliptic complex numbers with dual multiplication — a guided tour of W space

A research digest of Shuster and Köplinger (2010)

Abstract

This is a guided tour of the paper **J. A. Shuster and J. Köplinger, "Elliptic complex numbers with dual multiplication", *Appl. Math. Comput.* 216 (2010) 3497–3514** (DOI: 10.1016/j.amc.2010.04.069), which introduces *W space*: a two-dimensional real algebra built on a solution set of two basis elements (w) and $(-w)$, both primitive sixth roots of unity, whose integral and real powers trace out two mutually orthogonal elliptical orbits in the same $\mathbb{R}^2$ plane. Addition in W space is orbit-agnostic; multiplication is not, which means products must be resolved by a "context-sensitive" multiplication rule. This digest walks through the defining relations, the two-field structure, the dual norms, the unifying rules $\mathbb{W}_1$ and $\mathbb{W}_2$, and the signature "multi-star" fractal that arises when the orbit-ambiguity is iterated. The exposition is self-contained, cross-references follow-on work in the body, and takes the published journal paper as the authoritative reference.

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1 At a glance

W space is a two-dimensional real algebra in which the defining relation of ordinary complex conjugation — $\mathrm{conj}(z) + z = 0$ — is replaced by

$$\mathrm{conj}(z) + z = 1.$$

That single change identifies a solution set with **two** basis elements, written (w) and $(-w)$, and the rest of the structure follows from taking both of them seriously at once.

- **Two basis elements, both sixth roots of unity.** The defining relations identify a solution set $\{(w), (-w)\}$. Each element is a primitive sixth root of unity in its own field. They also happen to be additive inverses of each other — so under addition the algebra is a plain 2D real vector space, but under multiplication they behave as two independent anchors.
- **Two elliptical power-orbits, orthogonal to each other.** The integral and real powers of (w) trace out an ellipse; so do the powers of $(-w)$. Superimposed in the same $\mathbb{R}^2$, the long axes of the two orbit-ellipses are *perpendicular* — mirror-related across the real axis. See Figure 3.1 on page 3.

- **Addition is orbit-agnostic; multiplication is not.** Two points in $\mathbb{R}^2$ add as vectors regardless of which orbit they "came from". But a product depends on the orbit of each factor: in general $(w)^2 \neq (-w)^2$. To form a product one must first declare which orbit each factor belongs to. This is what the authors name a *context-sensitive multiplication.*
- **Two algebraically closed fields.** The subsystem $+\mathbb{W}$ generated by $\{1, (w)\}$ with its multiplication $\times$, and the subsystem $-\mathbb{W}$ generated by $\{1, (-w)\}$ with its multiplication $\circ$, are each a field isomorphic to $\mathbb{C}$. Cross-orbit products require a unifying rule beyond those two fields.
- **A signature fractal — the "multi-star".** Because squaring a point may take either the $\times$ or the $\circ$ branch, an N-fold iterated squaring generates up to 2^N possible values. Tracking the fraction that diverge yields a fractal whose multi-pointed star appearance is the visual fingerprint of the algebra. See Figures 7.1 and 7.2.

W space sits in a lineage of 2D hypercomplex systems (complex, split-complex, dual numbers). Its novelty is the simultaneous coexistence of two multiplication rules over a single real-2D vector space, rigorously reconstructed from a hypernumber intuition introduced — but never fully formalized — by Charles Musès [Muses1978] in the 1970s.

2 Starting point: what changes if $\operatorname{conj}(z) + z \neq 0$?

For ordinary complex numbers, two defining relations pin down the conjugate $\operatorname{conj}(z)$: the trace relation $\operatorname{conj}(z)+z = 0$ (on non-real axis elements), and the norm relation $\operatorname{conj}(z)\cdot z = 1 = z\cdot\operatorname{conj}(z)$ on the unit locus. Together these imply that the imaginary unit i must satisfy $i^2 = -1$, so the solution set of $z^2 = -1$ is $\{i, -i\}$ — and then $i^2 = (-i)^2 = -1$ identically.

Shuster and Köplinger [SK2010] ask: *what if we keep the norm relation but change the trace?* Specifically, they replace $\operatorname{conj}(z) + z = 0$ with

$$\operatorname{conj}(z) + z = 1. \tag{2.1}$$

Together with the unit-norm relation, this postulates a two-element solution set $\{(w), (-w)\}$ for the basis of the new algebra. Both elements are basis elements on equal footing — the name $(-w)$ is shorthand for "the other element of the solution set", not the coordinate-level negation of (w) in some pre-existing ambient space.

From these defining relations (Definition 2 of the paper) one derives:

- **Both (w) and $(-w)$ are sixth roots of unity.** In each case the minimal polynomial is $z^2 - z + 1 = 0$, i.e. $(w)^2 = (w) - 1$ and $(-w)^2 = (-w) - 1$. Real-power orbits land on an ellipse with six evenly spaced unit points.
- **$(w)^2 \neq (-w)^2$.** Computing directly, $(w)^2 = -1 + (w)$ and $(-w)^2 = -1 + (-w)$. Because (w) and $(-w)$ are not the same point, these two squares differ too. This is the characteristic signature of the algebra: what for ordinary complex numbers would be the identity $i^2 = (-i)^2$ *fails* in W space.
- **Two fields, two multiplications.** The subsystem $+\mathbb{W} := \{a + b(w)\}$ carries a complex-number-like multiplication, denoted $\times$. The subsystem $-\mathbb{W} := \{a + b(-w)\}$ carries its own complex-number-like multiplication, denoted $\circ$. Each is an algebraically closed field isomorphic to $\mathbb{C}$; they are *not* isomorphic as point sets in $\mathbb{R}^2$ — they are duals.

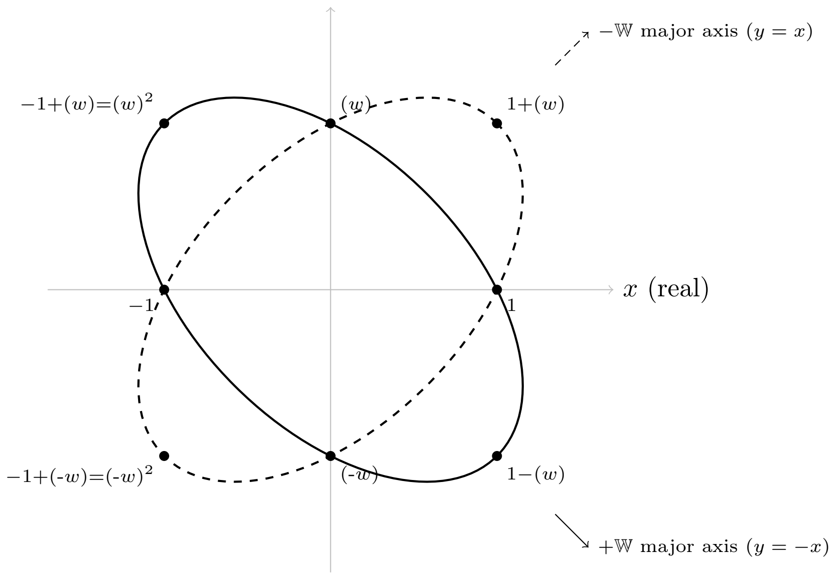

Fig. 3.1: The two power-orbits of W space, superimposed in a single $\mathbb{R}^2$ with Cartesian coordinates (x, y) in the basis $\{1, (w)\}$. Solid: $+\mathbb{W}$ unit ellipse $x^2 + xy + y^2 = 1$ with (w) at $(0, 1)$. Dashed: $-\mathbb{W}$ unit ellipse $x^2 - xy + y^2 = 1$ with $(-w)$ at $(0, -1)$. The two ellipses are mirror-related across the real axis; their long axes lie along $y = -x$ and $y = x$ respectively — mutually orthogonal. Redrawn freshly from Figures 1–3 of [SK2010].

- **Cross-orbit products need a unifying rule.** A product with one factor from $+\mathbb{W}$ and the other from $-\mathbb{W}$ is not defined by either field multiplication alone. The paper supplies the missing rule and calls the joint structure a *context-sensitive multiplication* (§4 below).

A subtlety flagged in the paper (§2.6): $(-w)$ *is* the additive inverse of (w) as a vector in $\mathbb{R}^2$ — that is, $(-w) + (w) = 0$ — but its role as the *other basis element of the solution set* does not reduce to coordinate negation. The paper is careful to keep these two roles distinct.

3 The two orbits in one plane

Here is the picture to keep in mind, drawn directly in Figure 3.1. In $\mathbb{R}^2$ viewed as the plane of W, with coordinates (x, y) in the basis $\{1, (w)\}$, there are two ellipses sharing the origin:

- the $+\mathbb{W}$ *power orbit*, the unit ellipse $x^2 + xy + y^2 = 1$, passing through the six unit points 1, (w), $-1 + (w)$, -1, $-(w)$, $1 - (w)$ — the integral powers $(w)^0 \ldots (w)^5$ — with $(w)^6 = 1$;
- the $-\mathbb{W}$ *power orbit*, the unit ellipse $x^2 - xy + y^2 = 1$, passing through the six unit points 1, $-(w)$, $-1 - (w)$, -1, (w), $1 + (w)$ — the integral powers $(-w)^0 \ldots (-w)^5$ expressed in the shared $\{1, (w)\}$ basis using $(-w) = -(w)$.

These two ellipses are **the same shape but differently oriented**. Their long axes point in perpendicular directions (along $y = -x$ and $y = x$ respectively). Every point $z \in \mathbb{R}^2$ belongs to both copies, but its *multiplicative* behavior depends on which side you are on. Two points add as ordinary vectors (addition is orbit-agnostic). To multiply two points, you must first declare the orbit of each: that choice commits the computation to one of the two multiplications, and to one of the two ellipses as the ambient unit locus. Cross-orbit products require the unifying rule of §4.

The paper works through the details — the derivation of all integral powers $(w)^n$ and $(-w)^n$, a substitution table for products across representations, and worked numerical examples — in §§2.4–2.6 of [SK2010].

4 Two fields and the unification

The paper introduces the idea of a *representation* (abbreviated "rep"): a label, either + or −, assigned to each element of W space, recording which of the two fields it belongs to. Within one representation, multiplication is the usual field multiplication. Across representations, something new is needed.

The unifying algebra is built from three auxiliary notions:

- rep(z) — the representation label of z (+, 0 for real, or −).
- **copoint** — the element with the same coordinates but the opposite representation.
- **dual point** — a specific reflection that, together with the copoint, closes the structure under multiplication.

With these in hand, the authors define two versions of the unifying multiplication:

- $\mathbb{W}_1$: the product takes its rep from the *right* factor;
- $\mathbb{W}_2$: the product takes its rep from the *left* factor.

Both versions are commutative, associative, and distributive. They are isomorphic but not identical: they represent two distinct choices of convention for resolving the cross-representation ambiguity. Either, consistently applied, yields a complete working algebra. The paper works primarily in $\mathbb{W}_1$.

The combined structure — the vector space $\mathbb{R}^2$ equipped with both representations and the chosen unification rule — is what the authors call **W space**.

5 Dual norms, dual solutions

Two striking consequences follow from the two-orbit structure.

Two norm values. The natural norm derived from the dual conjugate in W takes the form

$$\|z\|_+ = x^2 + xy + y^2 \qquad (\text{for } z \in +\mathbb{W}), \tag{5.1}$$

$$\|z\|_- = x^2 - xy + y^2 \qquad (\text{for } z \in -\mathbb{W}), \tag{5.2}$$

with (x, y) the same shared $\mathbb{R}^2$ coordinates in both cases. The unit level sets are the two ellipses of Figure 3.1. Their long axes lie along $y = -x$ and $y = x$ respectively, so they are **mutually orthogonal in** $\mathbb{R}^2$. The same coordinate pair (x, y) therefore has two distinct norm values depending on the orbit, and the same point in $\mathbb{R}^2$ lies on different-radius ellipses in $+\mathbb{W}$ and $-\mathbb{W}$. (This is why the algebra is called "elliptic complex": its natural metric is elliptic, not circular.)

Two solutions to every linear equation. Because the multiplication comes in two consistent forms — $\times$ for $+\mathbb{W}$, $\circ$ for $-\mathbb{W}$ — a linear equation $a \cdot z = b$ with fixed $a, b \in \mathbb{W}$ has two distinct solutions in general, one per orbit. For polynomial equations the situation compounds: a quadratic in W can have up to four distinct solutions (two roots $\times$ two orbits), reduced to fewer only by coincidence.

6 Variants and extensions

The authors explore several modifications, presented systematically in §6 of [SK2010]. The most important are:

- a *commutative multiplication* obtained by symmetrizing over the left-rep and right-rep conventions;
- a *minimum norm* that picks the smaller of the two rep-dependent norm values, yielding a single-valued but non-bilinear norm;
- the *joint space* $\mathbb{W}_{12}$, in which both $\mathbb{W}_1$ and $\mathbb{W}_2$ are carried simultaneously (requiring four components to track);
- *U space*, an octonionic-flavored generalization sketched as future work.

These are brief sections in the paper but signal the broader research program: W space is one stop in a systematic exploration of what happens when the defining symmetries of familiar number systems are deliberately relaxed.

7 The multi-star fractal

The signature visual artifact of the paper — the *"multi-star" W-space fractal* — is generated by exploiting the orbit-ambiguity of W-space multiplication directly.

Here is the construction (Appendix A of [SK2010]). Start with a point $A \in \mathbb{R}^2$. Squaring A in W space can be done in either orbit:

$$B_1 := A \times A \qquad \text{(square } A \text{ as a } +\mathbb{W} \text{ element)}, \tag{7.1}$$

$$B_2 := A \circ A \qquad \text{(square } A \text{ as a } -\mathbb{W} \text{ element)}. \tag{7.2}$$

One squaring step therefore yields **two** possible results, not one — the orbit is not fixed in advance. Squaring each of those two gives four:

$$C_{11} := B_1 \times B_1, \quad C_{12} := B_1 \circ B_1, \quad C_{21} := B_2 \times B_2, \quad C_{22} := B_2 \circ B_2.$$

After N squaring steps the starting point A has fanned out into up to 2^N possible results, one for every binary word of length N spelling out which of $\times$ or $\circ$ was used at each step.

For large N, each of those 2^N endpoints either "escapes" (its norm goes to infinity) or "collapses" (its norm goes to zero). One counts the fraction of the 2^N endpoints that collapse — call this the *convergence percentage* at A — and paints the pixel at A with a shade of gray:

- **black** — 100% convergence: every $\times/\circ$ path collapses; no orbit diverges;
- **white** — 0% convergence: every path diverges; no orbit collapses;

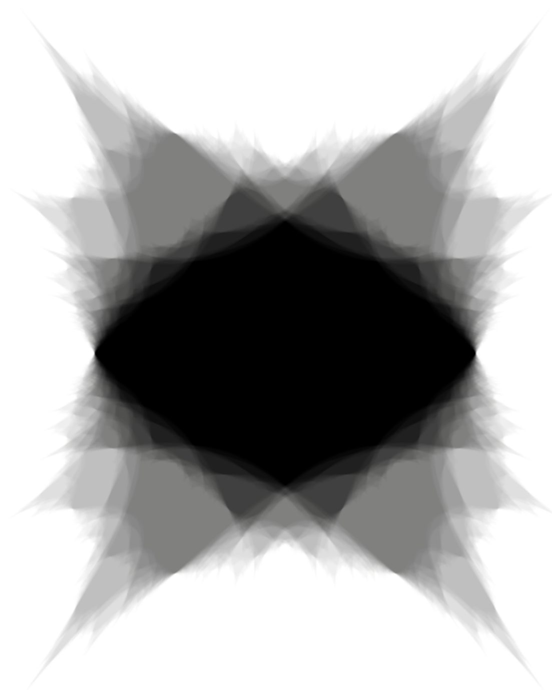

Fig. 7.1: Full multi-star fractal in the $\{[-2.1, +2.1], [-2.1, +2.1]\}$ parameter range of the $\{\text{real}, (w)\}$ plane, rendered at 5000×5000 pixels with iteration depth 25 and 766-gray palette. Image: J. Köplinger, *W Space – Multi-Star Fractal* gallery (reused under the author's CC-BY 4.0 re-licensing for this project from the original CC-BY-SA 3.0 posting on [KoeplWWW]). Source code: [WSpaceMandelbrot].

- **shades of gray** — partial convergence, a fraction of paths collapse, the remaining paths diverge.

Mapping this over a grid of starting points A gives the fractal shown in Figure 7.1. The name refers to the characteristic shape: multiple pointed arms radiating from the origin, with the solid "bulk" region (black — all paths collapse) concentrated near the center, and distinct gray-scale rays extending outward where partial divergence occurs. The fractal's richness comes entirely from the orbit ambiguity: the *same recursion applied inside ordinary* $\mathbb{C}$ (where multiplication is single-valued) gives nothing more than a solid black unit disk on a white background.

The $\text{Exp}(Z)$ fractal in Appendix B of [SK2010] uses a related construction: it forms the formal Taylor-series exponential of a W element by summing over *all* choices of $\times$ or $\circ$ at each power, producing a solution *set* whose points also exhibit fractal structure. A reference implementation is at [WSpaceMandelbrot]; further fractal galleries and an amplitude-modulated audio rendering are in the Musean hypernumbers section of the author's personal web page [KoeplWWW].

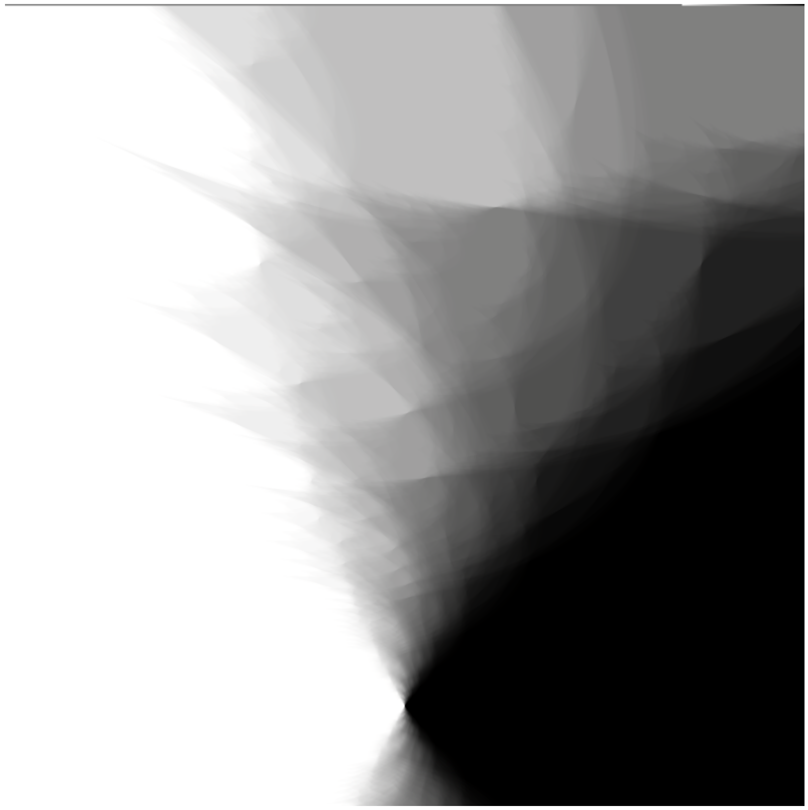

Fig. 7.2: Zoom into the left cusp of the multi-star, range $\{[-1.4, -0.6], [-0.1, +0.7]\}$. Same rendering parameters. Fine-grained self-similar structure along the $\times/\circ$ branching tree is visible at this scale. Image from the author's CC-BY 4.0 gallery [KoeplWWW].

8 Relationship to Musès' w numbers

Charles Arthur Musès (1919–2000) was an American philosopher and independent mathematician who, across a sequence of papers in *Applied Mathematics and Computation* between 1976 and 1994, proposed a family of extensions to the complex numbers under the name *hypernumbers* [Muses1976, Muses1978, Muses1980, Muses1994]. The program included "w numbers" (relevant here), "p and q numbers" (relevant to the companion paper [SK2011]), and several others. Musès earned his PhD from Columbia in 1951 with a dissertation on the 17th-century theosophists Böhme and Freher; he worked outside standard academic physics, edited the *Journal for the Study of Consciousness* (1968–1973), and is also associated with a spiritual practice he called the *Lion Path*. The German Wikipedia entry on Musès [WikiMusesDE] notes explicitly that "Köplinger und Shuster haben seine Ideen aufgegriffen und einige Hyperzahlen genauer analysiert" ("Köplinger and Shuster have taken up his ideas and analyzed several hypernumbers in detail").

In his hypernumbers-II paper [Muses1978] and elsewhere, Musès introduced a number w satisfying some but not all of the algebraic properties later formalized in [SK2010]. The intuition was that

w behaved like a sixth root of unity, and that the plane it generated had elliptic rather than circular geometry. Musès suggested applications in physics (spin, statistics of fermions) and cosmology.

Shuster and Köplinger [SK2010] credit Musès with the **preceding idea** — the sixth-root-of-unity intuition and the elliptic geometry — but document, with explicit citations to Musès' published work, that his formulation was inconsistent and underspecified:

- the algebra was never closed under a single consistent multiplication rule;
- the dual-representation phenomenon was not recognized, so the cross-representation products were ambiguous;
- some of Musès' derivations in specific examples are inconsistent with others in the same paper.

W space is offered as a clean, formal replacement that preserves what was right about Musès' intuition while giving it a consistent algebraic foundation. The paper includes a summary table (Table 2 of the preprint) listing which of Musès' claimed properties are recovered by W, which are recovered only in a specific representation, and which are not recovered at all.

An example paradox, and how it was resolved. A concrete illustration of what goes wrong in a naive single-multiplication reading of Musès' w numbers, and what the formal W-algebra fixes, is the following would-be theorem.

(Incorrect) Theorem. $(-w) \neq (-1) \cdot w$.

(Incorrect) Proof. Suppose $(-w) = (-1) \cdot w$. Squaring both sides gives

$$(-w)^2 = [(-1) \cdot w]^2, \qquad \text{i.e.} \qquad (-w) - 1 = (w) - 1,$$

hence $(-w) = (w)$, a contradiction. □

The proof fails because it silently uses two different multiplications: the left-hand squaring is the $\circ$ of $-\mathbb{W}$, while the right-hand squaring is the $\times$ of $+\mathbb{W}$. Each step is individually valid inside its own field, but composing them in one equation is not — the two sides live in different orbits. In the formal W algebra, a product (or a squaring) must commit to either $\times$ or $\circ$, and only after that commitment is equality of the two sides meaningful. Identifying precisely this kind of slip in informal Musean manipulation is what led the authors to articulate the context-sensitive multiplication rule.

This "advocate the intuition, critique specific claims, reconstruct" pattern recurs throughout the Shuster–Köplinger series.

9 Why "elliptic"?

The label *"elliptic complex numbers"* has two motivations:

1. **Geometric.** The level sets of the rep-dependent norm are ellipses in $\mathbb{R}^2$, not circles. The rotational symmetry of ordinary complex numbers is replaced by an affine rotation that preserves an ellipse.
2. **Arithmetic.** W contains a primitive *sixth* root of unity, not a fourth. The unit circle in ordinary complex numbers becomes a discrete hexagonal orbit in W.

The term is used here in a sense specific to this program and should not be confused with *elliptic curves* (cubic curves) or *elliptic functions* (doubly periodic meromorphic functions). The connection to those classical subjects, if any, is left to future work.

10 Applications hinted in the paper

The final section of [SK2010] (§7) points toward potential applications but stops short of specific physical predictions:

- **Spin-$\frac{1}{2}$ quantum systems and Stern–Gerlach experiments.** The two-rep structure of W has a natural pair-of-states interpretation. A formal analogy with the two outcomes of a spin measurement is noted, but no quantitative predictions are derived.
- **Carmody's results.** An appendix (C) reports numerical results on polar forms, exp/log, and the area of the W-unit ellipse, obtained by Kevin Carmody [Carmody] using an earlier informal formulation of the same algebra.

11 Reception in the literature

Richter 2021 — complex numbers related to semi-antinorms, ellipses or matrix homogeneous functionals. W.-D. Richter [Richter2021] constructs generalized complex numbers whose unit level sets may be ellipses, ℓ^p-shapes, or matrix-homogeneous contours. In §4.1 he locates his construction against W space by noting that in his algebra $z \odot z = (-z) \odot (-z)$, *while* $(w)^2 \neq (-w)^2$ in W. The two programs are thus **complementary**: Richter generalizes the shape of the norm level set but keeps squaring symmetric under sign-flip; W space keeps the level set an ellipse but splits squaring into two orbit-dependent branches.

Dündär 2024 — parabolic numbers. N. Dündär [Dundar2024] introduces a class of "parabolic numbers" in which the basis element satisfies $j^2 = p(y)$ for a specific coordinate-dependent function p. The paper cites W space [SK2010] in its bibliography as prior art, but does not structurally build on it; the construction is independent and motivated by a 2D Minkowski-isomorphic manifold with application to projectile motion and electron motion in a constant electric field.

Field-theoretic and algebra follow-ons. W space also appears as background in works on hyperbolic-octonion and sedenion electrodynamics (Demir, Tanışlı, Kansu and collaborators at Anadolu University; Mironov's "sedeons"; Weng; Chanyal), in pure-algebra studies of split/dual-quaternion variants, and in applications to cryptography (Thakur's KTRU/CSTRU families), neural networks (Popa), and signal processing (Snopek). These cite W space as part of a broader "non-standard 2D/4D/8D number system" reference set rather than as a specific structural input.

Own follow-on. The companion paper by the same authors on doubly nilpotent numbers [SK2011] applies the same methodology — critique of a Musèan intuition, formal reconstruction, accompanying fractal — to Musès' "p and q numbers". A separate digest covers that paper.

12 How to read the original

The paper was published in *Applied Mathematics and Computation* **216** (2010) 3497–3514, DOI: 10.1016/j.amc.2010.04.069. Personal versions of the authors' work are available on their personal pages and on ResearchGate; see [KoeplWWW, ResearchGateJK, ResearchGateJS].

The paper is organized as follows:

Section	Content
§1 Introduction	Motivation; contrast with circular complex numbers
§2 Circular vs elliptic	Defining relations DR-0/DR-1/DR-2; $+\mathbb{W}$ and $-\mathbb{W}$ fields
§3 Unification	$\mathbb{W}_1$, $\mathbb{W}_2$, rep function, copoint, dual point
§4 Geometric/algebraic	Dual norms, dual solutions to linear equations
§5 Musès w numbers	Comparison with Musès' earlier proposal
§6 Variants	Commutative form, min norm, $\mathbb{W}_{12}$, U space
§7 Summary	Applications hinted: spin, Stern–Gerlach
Appx. A	Multi-star fractal algorithm
Appx. B	Exp(Z) fractal algorithm
Appx. C	Carmody's polar/exp/log/ellipse-area results

13 About the authors

John A. Shuster is an independent researcher. His publications are available on ResearchGate [ResearchGateJS].

Jens Köplinger is the author of this digest. His personal page with publications, preprints, fractal galleries, and audio renderings is at [KoeplWWW]; his ResearchGate profile is [ResearchGateJK].

14 How to cite

For the original research, cite the published journal paper:

> J. A. Shuster and J. Köplinger, "Elliptic complex numbers with dual multiplication", *Appl. Math. Comput.* **216** (2010) 3497–3514. DOI: 10.1016/j.amc.2010.04.069.

For this digest, please refer to the ResearchGate entry on the author's profile [ResearchGateJK].

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