Non-associative decomposition of angular momentum operator using complex octonions.

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Non-associative multiplication

Algebra	$a(bc) = (ab)c$
• Real numbers	✓
• Complex numbers	✓
• Quaternions (4-dim.)	✓
• Matrix algebra	✓
• Octonions (8-dim.) [1]	generally not true

Research context

- Why physics on non-associative algebra?
  - Gives access to a hidden non-associative structure in quantum mechanics
    - Models parameters that are unobservable in principle, but do not violate Bell inequalities (no "hidden variables") [2-4]
    - Supersymmetric operator quantum mechanics [3,4]
  - Embeds known fundamental laws into formalisms that generally cannot be expressed in (associative) matrix algebra over complex numbers anymore.
    - Dirac equation with electromagnetic field [5,6]
    - Classical Maxwell equations [7-9]

General goals

- Aims at modeling known physical laws in more narrow formalisms, using fewer assumptions.
  - Split-octonions algebraically generate one time and three space dimensions, and tie Heisenberg uncertainty to a maximum speed (of light) [5,7].
  - Euclidean quantum gravity and electromagnetism in one closed formalism [6,10-12]
- Room for exploration:
  - Unobservability does not necessarily require formalisms to be indeterminate [3]. Philosophical impact on "hidden variables"?
  - Operator quantization of strongly interacting fields? [2,4]
  - Octonionic spinors describing fermion generations? [13,14]
  - Unification opportunity?

Research problem

- Previously known:
  - Non-associative decomposition of a supersymmetric momentum operator P^{μ} exists [4]:

$$P_\mu = \frac{1}{4}\sigma_\mu^{a\dot{a}}\{Q_a, Q_{\dot{a}}\}$$

- Needed to know:
  - Find non-associative decomposition of angular momentum operator $M^{\mu\nu}$.

Problem detail

Find totally antisymmetric angular momentum operator $M^{\mu\nu}$,

$$[R^\mu, R^\nu] = 2M^{\mu\nu},$$

Lorentz-invariant,

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\,(\eta^{\nu\rho} M^{\mu\sigma} + \eta^{\mu\sigma} M^{\nu\rho} - \eta^{\mu\rho} M^{\nu\sigma} - \eta^{\nu\sigma} M^{\mu\rho})\,,$$

generated by non-associative, closed components R^μ [4]:

$$(R^\mu, R^\nu, R^\rho) = (R^\mu R^\nu)\,R^\rho - R^\mu\,(R^\nu R^\rho) = 2\epsilon^{\mu\nu\rho\sigma} R_\sigma$$

Solution

Using complex octonions (conic sedenions) [15,16], $\mathbb{C} \otimes \mathbb{O}$, as:

$$b_{\mathbb{C}} := \{1, i_0\}$$

$$b_{\mathbb{O}} := \{1, i_1, \ldots, i_7\}$$

$$i_0 := i_n \epsilon_n \ (n \in \{1 \ldots 7\})$$

Basis for $M^{\mu\nu}$ and R^{μ} :

$$\{M^{\mu\nu}\} := \frac{1}{2} \begin{pmatrix} 0 & i_1 & i_2 & i_3 \\ -i_1 & 0 & -\epsilon_3 & \epsilon_2 \\ -i_2 & \epsilon_3 & 0 & -\epsilon_1 \\ -i_3 & -\epsilon_2 & \epsilon_1 & 0 \end{pmatrix}$$

$$\{R^0\} := \frac{i_4}{2}(1 + i_0),$$

$$\{R^1\} := \frac{i_5}{2}(1 - i_0),$$

$$\{R^2\} := \frac{i_6}{2}(1 - i_0),$$

$$\{R^3\} := \frac{i_7}{2}(1 - i_0).$$

Interpretation

- Immediate conclusions:
  - Operator quantum mechanics has a hidden, non-associative structure, because momentum and angular momentum operators can be decomposed as bilinear combination of some operators.
  - Yet this is not a “hidden variable” model, because non-associative elements are unobservable in principle.
  - Generally, provides justification for operator quantum mechanics on non-associative algebra.

Outlook

- The big question:
  - Is this "just" a philosophical point of view, or does it help description of physical law in new ways?
- Further investigation:
  - Do the R^μ have dynamical equations?
  - Is a non-associative decomposition of QFT possible?

Thank you

Thank you for attending this talk!

For details and notation cross reference, see:

“Supplement to 'Non-associative decomposition of angular momentum operator using complex octonions'”
(http://www.jenskoeplinger.com/P/SESAPS08_AngMomDecompSupp.pdf)

Appendix: Complex octonions

Quick notes about complex octonions (conic sedenions) [15,16]:

Modulus (norm):

$$|z| = |a + \sum b_n i_n + \sum c_n \varepsilon_n + d| := \sqrt[4]{(a^2 + b_n^2 - c_n^2 - d^2)^2 + 4(ad - b_n c_n)^2}$$

Multiplication table:

x	1	i_1	i_2	i_3	i_4	i_5	i_6	i_7	i_0	ε_1	ε_2	ε_3	ε_4	ε_5	ε_6	ε_7
1	1	i_1	i_2	i_3	i_4	i_5	i_6	i_7	i_0	ε_1	ε_2	ε_3	ε_4	ε_5	ε_6	ε_7
i_1	i_1	-1	i_3	$-i_2$	i_5	$-i_4$	$-i_7$	i_6	$-\varepsilon_1$	i_0	ε_3	$-\varepsilon_2$	ε_5	$-\varepsilon_4$	$-\varepsilon_7$	ε_6
i_2	i_2	$-i_3$	-1	i_1	i_6	i_7	$-i_4$	$-i_5$	$-\varepsilon_2$	$-\varepsilon_3$	i_0	ε_1	ε_6	ε_7	$-\varepsilon_4$	$-\varepsilon_5$
i_3	i_3	i_2	$-i_1$	-1	i_7	$-i_6$	i_5	$-i_4$	$-\varepsilon_3$	ε_2	$-\varepsilon_1$	i_0	ε_7	$-\varepsilon_6$	ε_5	$-\varepsilon_4$
i_4	i_4	$-i_5$	$-i_6$	$-i_7$	-1	i_1	i_2	i_3	$-\varepsilon_4$	$-\varepsilon_5$	$-\varepsilon_6$	$-\varepsilon_7$	i_0	ε_1	ε_2	ε_3
i_5	i_5	i_4	$-i_7$	i_6	$-i_1$	-1	$-i_3$	i_2	$-\varepsilon_5$	ε_4	$-\varepsilon_7$	ε_6	$-\varepsilon_1$	i_0	$-\varepsilon_3$	ε_2
i_6	i_6	i_7	i_4	$-i_5$	$-i_2$	i_3	-1	$-i_1$	$-\varepsilon_6$	ε_7	ε_4	$-\varepsilon_5$	$-\varepsilon_2$	ε_3	i_0	$-\varepsilon_1$
i_7	i_7	$-i_6$	i_5	i_4	$-i_3$	$-i_2$	i_1	-1	$-\varepsilon_7$	$-\varepsilon_6$	ε_5	ε_4	$-\varepsilon_3$	$-\varepsilon_2$	ε_1	i_0
i_0	i_0	$-\varepsilon_1$	$-\varepsilon_2$	$-\varepsilon_3$	$-\varepsilon_4$	$-\varepsilon_5$	$-\varepsilon_6$	$-\varepsilon_7$	-1	i_1	i_2	i_3	i_4	i_5	i_6	i_7
ε_1	ε_1	i_0	ε_3	$-\varepsilon_2$	ε_5	$-\varepsilon_4$	$-\varepsilon_7$	ε_6	i_1	1	$-i_3$	i_2	$-i_5$	i_4	i_7	$-i_6$
ε_2	ε_2	$-\varepsilon_3$	i_0	ε_1	ε_6	ε_7	$-\varepsilon_4$	$-\varepsilon_5$	i_2	i_3	1	$-i_1$	$-i_6$	$-i_7$	i_4	i_5
ε_3	ε_3	ε_2	$-\varepsilon_1$	i_0	ε_7	$-\varepsilon_6$	ε_5	$-\varepsilon_4$	i_3	$-i_2$	i_1	1	$-i_7$	i_6	$-i_5$	i_4
ε_4	ε_4	$-\varepsilon_5$	$-\varepsilon_6$	$-\varepsilon_7$	i_0	ε_1	ε_2	ε_3	i_4	i_5	i_6	i_7	1	$-i_1$	$-i_2$	$-i_3$
ε_5	ε_5	ε_4	$-\varepsilon_7$	ε_6	$-\varepsilon_1$	i_0	$-\varepsilon_3$	ε_2	i_5	$-i_4$	i_7	$-i_6$	i_1	1	i_3	$-i_2$
ε_6	ε_6	ε_7	ε_4	$-\varepsilon_5$	$-\varepsilon_2$	ε_3	i_0	$-\varepsilon_1$	i_6	$-i_7$	$-i_4$	i_5	i_2	$-i_3$	1	i_1
ε_7	ε_7	$-\varepsilon_6$	ε_5	ε_4	$-\varepsilon_3$	$-\varepsilon_2$	ε_1	i_0	i_7	i_6	$-i_5$	$-i_4$	i_3	i_2	$-i_1$	1

Other relations:

$$e^{\varepsilon_n \alpha} = \cosh\ \alpha + \varepsilon_n(\sinh\ \alpha)$$

$$\ln \varepsilon_n = \frac{\pi}{2}(i_0 - i_n)$$

$$\varepsilon_n^\alpha = \frac{1}{2}[(1 - \varepsilon_n) + (1 + \varepsilon_n)e^{-\pi i_n \alpha}]$$

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