Supplement to "Non-associative decomposition of angular momentum operator using complex octonions"

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This supplement to the SESAP 2008 talk (in Raleigh, NC, USA; SES08-2008-000089) details the shown algebraic representation per V. Dzhunushaliev, arXiv:0805.3221, "A hidden nonassociative structure in quantum mechanics", equations (17) through (20), using complex octonions $\mathbb{C} \otimes \mathbb{O} \equiv \mathbb{O} \oplus \mathbb{O}$.

A. Definition of complex octonions

"Complex octonions" are octonions with complex coefficients. They are synonymous with "conic sedenions" in some other publications. Here the octonion basis $b_{\mathbb{O}} := \{1, i_1, \ldots, i_7\}$ is chosen, with

$$i_m i_n \;=\; a_{mnl} i_l - \delta_{mn} \tag{1}$$

and totally antisymmetric a_{mnl} and

$$\begin{aligned} a_{mnl} \;&=\; +1 && (2)\\ \text{for } mnl \;&\in\; \{123, 145, 176, 246, 257, 347, 365\}\,. && (3) \end{aligned}$$

The complex number coefficients are written to basis $b_{\mathbb{C}} := \{1, i_0\}$ where $i_0^2 = -1$ and i_0 multiplication is associative, commutative, and distributive with all other basis elements. The product of i_0 and i_n is written

$$i_0 i_n = -\epsilon_n. \tag{4}$$

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B. Decomposition rules

The following relations will now be modeled using complex octonions:

$$\begin{aligned} \{R^\mu, R^\nu\} \;&=\; 0, && \text{(B1)}\\ [R^\mu, R^\nu] \;&=\; 2M^{\mu\nu}, && \text{(B2a)}\\ [M^{\mu\nu}, M^{\rho\sigma}] \;&=\; i_0\left(\eta^{\nu\rho}M^{\mu\sigma} + \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\sigma}M^{\mu\rho}\right), && \text{(B3)}\\ (R^\mu, R^\nu, R^\rho) = (R^\mu R^\nu)\,R^\rho - R^\mu\,(R^\nu R^\rho) \;&=\; 2\varepsilon^{\mu\nu\rho\sigma}R_\sigma, && \text{(B4a1)}\\ (R^\mu, R^\nu, R^\rho) \;&=\; -\left(R^\rho, R^\nu, R^\mu\right). && \text{(B4a2)} \end{aligned}$$

We're using $\eta^{\nu\rho} = diag\,(1, -1, -1, -1)$.

C. Modeling the $M^{\mu\nu}$

Per relation (B2a), the $M^{\mu\nu}$ must be totally anti-symmetric, which determines $M^{\mu\mu} = 0$ and $M^{\mu\nu} = -M^{\nu\mu}$. That leaves six independent parameters, which will be chosen as:

$$M^{\mu\nu} \;:=\; \frac{1}{2}\begin{pmatrix} 0 & i_1 & i_2 & i_3 \\ -i_1 & 0 & -\epsilon_3 & \epsilon_2 \\ -i_2 & \epsilon_3 & 0 & -\epsilon_1 \\ -i_3 & -\epsilon_2 & \epsilon_1 & 0 \end{pmatrix}, \tag{5}$$

with $i_0 i_1 = -\epsilon_1$ and similar.

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Now, testing relation (B3):

1. If $M^{\mu\nu} = M^{\rho\sigma}$ then the commutator is zero (trivial).
2. If $\mu = \nu$ or $\rho = \sigma$, then either $M^{\mu\nu}$ or $M^{\rho\sigma}$ must be zero $\Longrightarrow$(B3) tests OK.
3. If all four $\{\mu, \nu, \rho, \sigma\}$ are different, then $M^{\mu\nu}$ must be $\pm i_0 M^{\rho\sigma}$:

$$M^{01} = \frac{i_1}{2} = \frac{i_0\epsilon_1}{2} = -i_0 M^{23}, \tag{6}$$
$$M^{02} = \frac{i_2}{2} = \frac{i_0\epsilon_2}{2} = i_0 M^{13}, \tag{7}$$
$$M^{03} = \frac{i_3}{2} = \frac{i_0\epsilon_3}{2} = -i_0 M^{12}, \tag{8}$$

   But, because i_0 is commutative, the commutator in (B3) must be zero as well, as required. The remaining combinations are permutations of these three relations $\Longrightarrow$(B3) tests OK.

4. That leaves only the following combinations, and trivial permutations thereof:

$$\left[M^{01}, M^{02}\right] = \frac{i_3}{2} = \frac{i_0\epsilon_3}{2} = -i_0\eta^{00} M^{12}, \tag{9}$$
$$\left[M^{01}, M^{03}\right] = -\frac{i_2}{2} = -\frac{i_0\epsilon_2}{2} = -i_0\eta^{00} M^{12}, \tag{10}$$
$$\left[M^{12}, M^{13}\right] = -\frac{i_1}{2} = -\frac{i_0\epsilon_1}{2} = -i_0\eta^{11} M^{23}, \tag{11}$$
$$\left[M^{10}, M^{13}\right] = -\frac{\epsilon_3}{2} = \frac{i_0 i_3}{2} = -i_0\eta^{11} M^{03}, \tag{12}$$
$$\left[M^{20}, M^{21}\right] = -\frac{\epsilon_1}{2} = \frac{i_0 i_1}{2} = -i_0\eta^{22} M^{01}, \tag{13}$$
$$\left[M^{20}, M^{23}\right] = -\frac{\epsilon_3}{2} = \frac{i_0 i_3}{2} = -i_0\eta^{22} M^{03}, \tag{14}$$
$$\left[M^{30}, M^{31}\right] = -\frac{\epsilon_1}{2} = \frac{i_0 i_1}{2} = -i_0\eta^{33} M^{01}, \tag{15}$$
$$\left[M^{30}, M^{32}\right] = -\frac{\epsilon_2}{2} = \frac{i_0 i_2}{2} = -i_0\eta^{33} M^{02}. \tag{16}$$

   All remaining combinations are obtained by either by switching the arguments of the commutator, or by switching the indices of the $M^{\mu\nu}$. Because the commutator and also the $M^{\mu\nu}$ are anti-symmetric $\Longrightarrow$(B3) tests OK.

D. Modeling the R^μ

Having (B3) satisfied, the remaining R^μ are modeled using commutative and associative factors a^μ:

$$R^0 := a^0 i_4, \tag{17}$$
$$R^1 := a^1 i_5, \tag{18}$$
$$R^2 := a^2 i_6, \tag{19}$$
$$R^3 := a^3 i_7. \tag{20}$$

The R^μ are therefore modeled from the anti-commutative, anti-associative four-tuple $\{i_4, i_5, i_6, i_7\}$ from the octonions, and supplied with a commutative and associative factor a^μ. This way, they immediately satisfy relation (B1) and (B4a-2).

For relation (B2a), $[R^\mu, R^\nu] = 2M^{\mu\nu}$, we have:

$$\left[R^0, R^1\right] = 2a^0 a^1 i_1 = 4a^0 a^1 M^{01}, \tag{21}$$
$$\left[R^0, R^2\right] = 2a^0 a^2 i_2 = 4a^0 a^2 M^{02}, \tag{22}$$
$$\left[R^0, R^3\right] = 2a^0 a^3 i_3 = 4a^0 a^3 M^{03}, \tag{23}$$

$$\left[R^1, R^2\right] = -2a^1a^2i_3 = -2a^1a^2i_0\epsilon_3 = 4a^1a^2i_0M^{12}, \tag{24}$$
$$\left[R^1, R^3\right] = 2a^1a^3i_2 = 2a^1a^3i_0\epsilon_2 = 4a^1a^3i_0M^{13}, \tag{25}$$
$$\left[R^2, R^3\right] = -2a^2a^3i_1 = -2a^2a^3i_0\epsilon_1 = 4a^2a^3i_0M^{23}. \tag{26}$$

The relations between the indices of R^μ and $M^{\mu\nu}$ are already correct, lastly the (commutative, associative) $a^\mu a^\nu$ products will be adjusted.

The products a^0a^j $(j = 1, 2, 3)$ don't have to compensate a factor i_0, whereas all other products a^ja^k $(j, k = 1, 2, 3$ and $j \neq k)$ have to compensate a factor i_0. We can therefore chose:

$$a^0 := \sqrt{\frac{i_0}{2}}, \tag{27}$$
$$a^j := \frac{1}{\sqrt{2i_0}} \qquad (j = 1, 2, 3)\,, \tag{28}$$
$$\text{with } \sqrt{i_0} = \frac{1}{\sqrt{2}}\left(1 + i_0\right), \tag{29}$$
$$\frac{1}{\sqrt{i_0}} = \frac{1}{\sqrt{2}}\left(1 - i_0\right). \tag{30}$$

That satisfies relation (B2a).

With all R^μ determined, the remaining relation (B4a-1) needs to be checked. Because the R^μ are alternative, but anti-associative and anti-commutative, we can confirm that the associator (R^μ, R^ν, R^σ) is totally anti-symmetric as required. The product of three factors (using $R^0 = R_0$ and $R^i = -R_i$) is:

$$\left(R^0R^1\right)R^2 = 2a^0a^1a^2\left(i_1i_6\right) = \frac{1}{\sqrt{2i_0}}\left(-i_7\right) = \varepsilon^{0123}R_3, \qquad \left(\varepsilon^{0123} = +1, \eta^{33} = -1\right) \tag{31}$$
$$\left(R^0R^1\right)R^3 = 2a^0a^1a^3\left(i_1i_7\right) = \frac{1}{\sqrt{2i_0}}i_6 = \varepsilon^{0132}R_2, \qquad \left(\varepsilon^{0132} = -1, \eta^{22} = -1\right) \tag{32}$$
$$\left(R^0R^2\right)R^3 = 2a^0a^2a^3\left(i_2i_7\right) = \frac{1}{\sqrt{2i_0}}\left(-i_5\right) = \varepsilon^{0231}R_1, \qquad \left(\varepsilon^{0231} = +1, \eta^{11} = -1\right) \tag{33}$$
$$\left(R^1R^2\right)R^3 = -2a^1a^2a^3\left(i_3i_7\right) = \frac{1}{\sqrt{2i_0^3}}i_4 = -\sqrt{\frac{i_0}{2}}i_4 = \varepsilon^{1230}R_0. \qquad \left(\varepsilon^{1230} = -1, \eta^{00} = 1\right) \tag{34}$$

The last line used $1/\sqrt{i_0^3} = \sqrt{i_0}/\sqrt{i_0^4} = -\sqrt{i_0}$.

E. Result summary

We can model relations (B1), (B2a), (B3), and (B4a-1/2) from above through complex octonions (conic sedenions) as:

$$M^{\mu\nu} := \frac{1}{2}\begin{pmatrix} 0 & i_1 & i_2 & i_3 \\ -i_1 & 0 & -\epsilon_3 & \epsilon_2 \\ -i_2 & \epsilon_3 & 0 & -\epsilon_1 \\ -i_3 & -\epsilon_2 & \epsilon_1 & 0 \end{pmatrix}, \tag{35}$$
$$R^0 := \frac{i_4}{2}\left(1 + i_0\right), \tag{36}$$
$$R^j := \frac{i_{j+4}}{2}\left(1 - i_0\right). \tag{37}$$

Minkowski metric was chosen $\eta^{\nu\rho} = diag\left(1, -1, -1, -1\right)$.

F. Addendum: Isomorphisms

Split-octonions are a subalgebra of the complex octonions, and several different notations exist. This sections quickly cross-references isomorphisms. In the notation here, a split-octonion basis can be chosen e.g. as:

$$b_{\text{split-}\mathbb{O}} := \{1, i_1, i_2, i_3, \epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7\}\,.$$

This maps directly to notation chosen by Merab Gogberashvili (e.g. arXiv:0808.2496) as:

$$b_{\text{split-}\mathbb{O}} \equiv \{1, j_1, j_2, j_3, I, J_1, J_2, J_3\}\,.$$

The following notation is also used at times, based on quaternions $b_{\mathbb{Q}} := \{1, i, j, k\}$ and non-real basis element l with $l^2 = 1$:

$$b_{\text{split-}\mathbb{O}} \equiv \{1, i, j, k, -l, -li, -lj, -lk\}\,.$$

In "Zorn's vector matrix" algebra, this basis is isomorphic to:

$$\begin{aligned} b_{\text{split-}\mathbb{O}} \;\equiv\; & \left\{ \begin{bmatrix} 1 & (0,0,0) \\ (0,0,0) & 1 \end{bmatrix}, \begin{bmatrix} 0 & (1,0,0) \\ (-1,0,0) & 0 \end{bmatrix}, \begin{bmatrix} 0 & (0,1,0) \\ (0,-1,0) & 0 \end{bmatrix}, \begin{bmatrix} 0 & (0,0,1) \\ (0,0,-1) & 0 \end{bmatrix}, \right. \\ & \left. \begin{bmatrix} -1 & (0,0,0) \\ (0,0,0) & 1 \end{bmatrix}, \begin{bmatrix} 0 & (-1,0,0) \\ (-1,0,0) & 0 \end{bmatrix}, \begin{bmatrix} 0 & (0,-1,0) \\ (0,-1,0) & 0 \end{bmatrix}, \begin{bmatrix} 0 & (0,0,-1) \\ (0,0,-1) & 0 \end{bmatrix} \right\}. \end{aligned}$$

Note that Zorn's vector matrix product is defined differently from regular matrix algebra.