

# Doubly nilpotent numbers in the 2D plane

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## Abstract

Dual numbers, split-quaternions, split-octonions, and other number systems with nilpotent spaces have received sporadic yet persistent interest, beginning from their roots in the 19th century, to more recent attention in connection with supersymmetry in physics. In this paper, a number system in the 2D plane is investigated, where the squares of its basis elements  $p$  and  $q$  each map into the coordinate origin. Modeled similar to an original concept by C. Musès, this new system will be termed “PQ space” and presented as a generalization of nilpotence and zero. Compared to the complex numbers, its multiplicative group and underlying vector space are equipped with as little as needed modifications to achieve the desired properties. The locus of real powers of basis elements  $p^\alpha$  and  $q^\alpha$  resembles a four-leaved clover, where the coordinate origin at  $(0,0)$  will not only represent the additive identity element, but also a map of “directed zeroes” from the multiplicative group. Algebraic and geometric properties of PQ space are discussed, and its naturalness advertised by comparison with other systems. The relation to Musès’ “ $p$  and  $q$  numbers” is shown and its differences defended. Next to possible applications and extensions, a new butterfly-shaped fractal is generated from a recursion algorithm of Mandelbrot type.

*Key words:* Nilpotent number, generalized nilpotence, vector space, duality, fractal, hypernumber, projective map, projection, PQ space

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## 1 Introduction

Nilpotent number systems contain one or more basis elements  $u$  that vanish when raised to some positive integral power  $n$ , i.e.  $u^n = 0$ . The dual numbers may be one of the better known examples, forming a two dimensional vector space to a basis  $b_{\text{dual}} := \{1, u\}$ . Other algebras that contain nilpotent spaces include split-quaternions, split-octonions, as well as quaternions or octonions with complex coefficients (see e.g. [1] for some points and references). The more recent interest in nilpotence in the applied sciences, namely in physics, has focused on more or less exotic operator spaces over manifolds.

In this paper, we will examine how nilpotence could be generalized algebraically, in a way that provides new means of manipulation; namely, to allow not only for addition and multiplication, but also for division and exponentiation along axes of generalized nilpotent basis elements. The speculation is that such a system could be used at some point, to model natural law not necessarily on operator spaces, but using algebraic systems that more closely represent the behavior under investigation.

The selected approach here will define “PQ space” in the two dimensional plane, with homomorphic maps of a general point in  $\{p, q\}$  into an additive group,  $\text{PQ}^+$ , and a multiplicative group with zero,  $\text{PQ}^\times$ . Taken separately,  $\text{PQ}^+$  corresponds to a vector space, and  $\text{PQ}^\times$  to multiplication in the complex numbers. The morphisms are chosen such that the square of each basis element maps into the coordinate origin, namely  $p^2 \mapsto (0,0)$  and  $q^2 \mapsto (0,0)$ . This is understood as a generalization of nilpotence, as it relaxes equality “ $p^2 = 0$ ”.

Algebraic and geometric properties of this system will be examined, and compared to a system brought forward by Charles Musès [2,3,4,5,6], “ $p$  and  $q$  numbers”, as part of his hypernumbers concept.

Possible applications of such a system might include supersymmetry in physics, since both basis elements of PQ have anticommutator relations  $\{p, p\} = \{q, q\} = 0$ , therefore hinting at algebraic generators of supersymmetry. Also, extensions of the previous W space type [7] can be envisioned. A fractal algorithm of Mandelbrot type yields a new, butterfly-shaped fractal.

## 2 Investigating possible generalizations of nilpotence

Before defining PQ space as doubly nilpotent numbers in the 2D plane, this section will sketch certain aspects of what one may understand as a generalization of nilpotence, and possible motivation for keeping certain aspects of a classical concept invariant while modifying others.

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## 2.1 Multiplication and division in the dual numbers

Dual numbers can be represented as a two dimensional vector space, in the  $(x, y)$  plane to a basis  $b_{\text{dual}} := \{1, u\}$  with vector multiplication between two points  $A := (x_A, y_A)$ ,  $B := (x_B, y_B)$ ,  $x_A, y_A, x_B, y_B \in \mathbb{R}$ , defined as:

$$A \cdot B := (x_A x_B, x_A y_B + y_A x_B), \quad (1)$$

$$A \div B = \left( \frac{x_A}{x_B}, \frac{y_A x_B - x_A y_B}{x_B^2} \right) \quad \text{for } x_B \neq 0. \quad (2)$$

Division ( $\div$ ) is undefined for any  $B$  along the  $u$ -axis, where

$$(y_B u)^2 = (0, y_B) \cdot (0, y_B) = (0, 0) \equiv "0". \quad (3)$$

Similarly, there is no square root of  $u$ , or any other real powers of  $u$  other than  $u^1 = u$  and  $u^n = 0$  for every  $n \in \{2, 3, \dots\}$ .

## 2.2 Real powers of a nilpotent basis element: power orbit

For a possible generalization of nilpotence, we will now explore what it would take for a nilpotent basis element  $u$  to have a power orbit, i.e., a continuous nondegenerate locus of real powers of  $u$ . Negative powers would implicitly provide a means of division along this  $u$ -axis as well, and therewith provide a richer arithmetic as compared to the dual numbers.

The chosen constraints are:

- (1) Vector space: The system will contain a two dimensional vector space over the reals. This defines addition between two points,  $A + B$ , as well as a scalar multiplication.
- (2) Generalized nilpotency: At least one basis element  $u$  must exist where the square of all points  $A$  along the  $u$ -axis falls into the coordinate origin, i.e.:

$$A^2 \mapsto (0, 0) \text{ for all } A := y_A u, y_A \in \mathbb{R}. \quad (4)$$

This relaxes equality " $A^2 = 0$ " by allowing a map between  $A^2$  and the  $(0, 0)$  point.

- (3) Continuous power orbit: A power orbit is defined as the locus of all real powers of  $u$ , and required to be continuous, nondegenerate, with unit element

$$u^1 := u, \quad (5)$$

and mapping all even powers of  $u$  into the coordinate origin:

$$u^{2n} \mapsto (0, 0) \text{ for all } n \in \mathbb{Z}. \quad (6)$$

- (4) Similarity to the complexes  $\mathbb{C}$ : Multiplication between any two numbers should be defined as similar as possible to multiplication in the complexes: Each number is represented by a modulus (a "length-type" measure) and a cyclic angle with period  $2\pi$ . The modulus of the product will then be the product of the individual moduli, and the resulting angle is the sum of the individual angles:

$$|A \cdot B| = |A| |B|, \quad (7)$$

$$\angle (AB) = \angle (A) + \angle (B). \quad (8)$$

- (5) Radial symmetry of modulus function: For two points  $A, B$  with  $|A| = |B|$ , the following expressions are equivalent:

$$\angle (A) = \angle (B) + \pi \iff A = -B. \quad (9)$$

In other words, multiplication in the complex numbers will be modified by generalizing circles of isonorm  $\|A\| = x_A^2 + y_A^2$  to power orbits of isomodulus  $|A| := f(x_A, y_A)$ , while maintaining as many other algebraic properties of the complexes as possible.

Figure 1. Possible power orbits (sketch) for generalized nilpotent numbers.

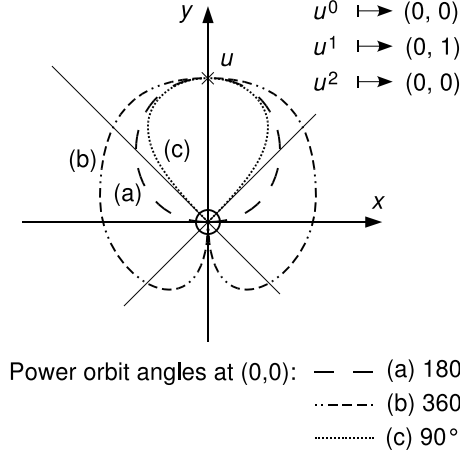
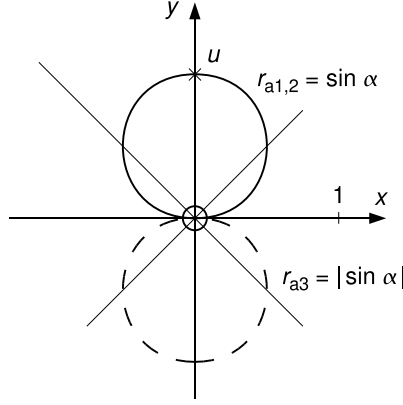


Figure 2. Power orbit examples for  $r = \sin \alpha$  and  $r = |\sin \alpha|$ .



### 2.3 Considerations for power orbit selection

The desire to model a generalization of nilpotence with “as little as possible” modifications as compared to the complex numbers is obviously vague, as it is lacking specifics about what exactly may be subject to modification, and what is not. This section will demonstrate a few examples of choices for power orbit, to support the argument for naturalness of our ultimate choice.

In the two dimensional plane, figure 1 gives some basic examples of a continuous power orbit of  $u$ , which transition through the origin  $(0,0)$  for  $u^0$  and  $u^2$ , and  $(0,1)$  for  $u^1$ .

Power orbit (a) indicates a smooth transition through both  $(0,0)$  and  $(0,1)$ , and could be modeled using radius relations

$$r_{a1} := \sin \alpha \quad \text{with } 0 \leq \alpha < \pi, \quad (10)$$

$$r_{a2} := \sin \alpha \quad \text{with } \alpha \in \mathbb{R}, \text{ or} \quad (11)$$

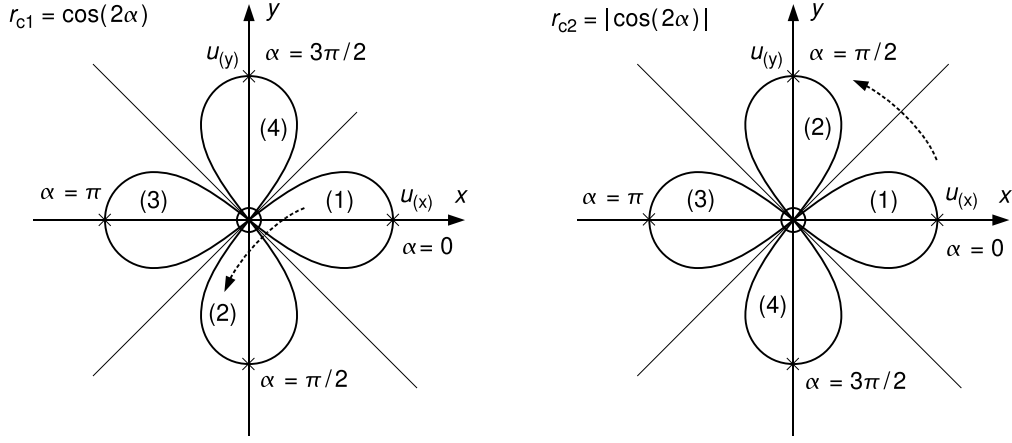
$$r_{a3} := |\sin \alpha| \quad \text{with } \alpha \in \mathbb{R}. \quad (12)$$

As indicated in figure 2, radius relation  $r_{a1}$  would allow for uniquely identifying each point  $A = (x_A, y_A)$  in the top half plane ( $y_A > 0$ ) using the two parameters modulus  $|A|$  and angle  $\alpha$ , but would require to restrict the parameter range of the sine function to the interval  $0 \leq \alpha < \pi$ . Radius relation  $r_{a2}$  would allow any real angle  $\alpha$ , however, would cover the top plane twice per period  $0 \dots 2\pi$ , as  $r_{a2}$  becomes negative for  $\pi < \alpha < 2\pi$ . Radius relation  $r_{a3}$  would allow to cover the top and bottom half-planes, ( $y_A > 0$ ) and ( $y_A < 0$ ), however, the horizontal axis ( $x_A, 0$ ) with  $x_A \neq 0$  would still not have a possible representation through modulus and angle.

In all three cases, it is dissatisfying that the horizontal axis cannot lie on a power orbit (with the exception of the coordinate origin).

A similar situation exists in a power orbit from figure 1, shape (b), e.g.:

Figure 3. Power orbit examples for  $r_{c1} = \cos(2\alpha)$  and  $r_{c2} = |\cos(2\alpha)|$ .



$$r_b := \frac{1}{2} (1 + \sin \alpha) \quad \text{with } \alpha \in \mathbb{R}. \quad (13)$$

Here, the negative  $u$ -axis  $(0, y_A)$  with  $y_A < 0$  cannot be mapped to a modulus, in addition to lack of the desired symmetry through the origin.

Power orbits from figure 1, shape (c) appear to be a natural candidate, to satisfy arithmetic along both axes which generate the two dimensional vector space. Radius functions

$$r_{c1} := \cos(2\alpha), \quad \text{or} \quad (14)$$

$$r_{c2} := |\cos(2\alpha)|, \quad (15)$$

with  $\alpha \in \mathbb{R}$  generate what resembles a double figure eight, or four-leaved clover shape. As shown in figure 3, all points in the plane except the diagonals can be expressed through a unique angle and modulus, by scaling the power orbit radius function. In each interval,  $0 \leq \alpha < 2\pi$ , the power orbit maps four times into the coordinate origin:

$$r_{c1}(\alpha) = r_{c2}(\alpha) = 0 \text{ for } \alpha \in \left\{ \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \right\}, \quad (16)$$

and therefore satisfies the constraints from section 2.2, namely by identifying

$$u^{2n} \mapsto (0, 0), \quad (17)$$

to correspond to angles:

$$\alpha = \frac{\pi}{4} (2n + 1), \quad n \in \mathbb{Z}. \quad (18)$$

Such a system would therefore resemble a ‘‘doubly nilpotent’’ space.

As for naturalness or simplicity, the radius function  $r_{c1}$  has the property of being continuous and differentiable in the two dimensional plane, however, resolves to a negative radius function for  $\frac{\pi}{4} < \alpha < \frac{3\pi}{4}$  and  $\frac{5\pi}{4} < \alpha < \frac{7\pi}{4}$ . Conversely,  $r_{c2}$  is always positive, but not differentiable at the coordinate origin. Either way, a choice must be made.

Figure 3 also represents the power orbit first suggested for the ‘‘ $p$  and  $q$  number’’ concept by Charles Musès (e.g. [2,3]), which will be discussed in detail later (section 4). Later, he proposed a different shape [4,5,6] (figure 4):

$$r_{c3} := \sqrt{2} \sin(2\alpha) \cos \alpha = 2\sqrt{2} \sin \alpha \cos^2 \alpha. \quad (19)$$

Next to the diagonals  $x = \pm y$ , no point in the bottom-left half of the plane ( $x + y < 0$ ) could possess a modulus from scaling this power orbit. Similar to figure 2, the radius  $r_{c3}$  is negative for  $\pi < \alpha < 2\pi$  and therefore retraces the two leaves generated for  $0 < \alpha < \pi$ . Choosing the absolute of the radius,  $r_{c4} := |r_{c3}|$ , can restore symmetry through the origin. There, a modulus exists for all points except where  $y = \pm x$ .

Figure 4. A power orbit used by C. Musès for “ $p$  and  $q$ ” numbers, together with  $|r_{c3}|$ .

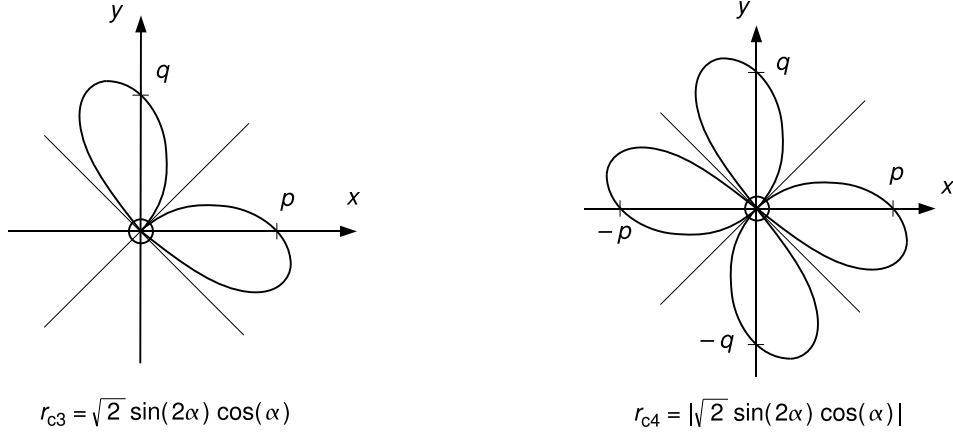


Table 1

Candidate  $r(\alpha)$  and lines of isomodulus (power orbits) for generalized nilpotence.

Radius function $r(\alpha)$	Figure	Isomodulus $f(x, y) = const.$	Properties
n/a	n/a	$ x $	Dual numbers: no division for $x = 0$ ; no real powers for $y \neq 0$ .
$r_{a1,2} = \sin \alpha$	2	$(x^2 + y^2)/y$	No point $y \leq 0$ expressible through a modulus.
$r_{a3} =  \sin \alpha $	2	$(x^2 + y^2)/ y $	No modulus for points $y = 0$ .
$r_b = \frac{1}{2}(1 + \sin \alpha)$	1	$2(x^2 + y^2)/(y + \sqrt{x^2 + y^2})$	No modulus for $(0, y)$ where $y < 0$ ; no symmetry through $(0, 0)$ .
$r_{c1} = \cos(2\alpha)$	3	$(x^2 + y^2)^{3/2}/ x^2 - y^2 $	No modulus along diagonals $x = \pm y$ , negative radius possible.
$r_{c2} =  \cos(2\alpha) $	3	$(x^2 + y^2)^{3/2}/ x^2 - y^2 $	No modulus for $x = \pm y$ , not differentiable at $(0, 0)$ .
$r_{c3} = \sqrt{2} \sin(2\alpha) \cos \alpha$	4	$(x^2 + y^2)^2/[(x+y)(x-y)^2]$	No modulus for $x + y \leq 0$ .
$r_{c3} =  \sqrt{2} \sin(2\alpha) \cos \alpha $	4	$(x^2 + y^2)^2/[ x+y (x-y)^2]$	No modulus for $x = \pm y$ , not differentiable at $(0, 0)$ .

#### 2.4 Simplicity and symmetry as arguments for a preferred system

The candidate spaces sketched in this section, as possibilities of generalizing nilpotence, are summarized in table 1. All have in common that they specify an isomodulus  $f(x, y)$  and radius  $r(x, y)$ , with its associated radius function  $r(\alpha)$  using Cartesian angle  $\alpha$ , i.e.:

$$r(x, y) = \sqrt{x^2 + y^2} = r(\alpha) f(x, y). \quad (20)$$

This list is certainly not comprehensive or complete, and selection of one system over another requires further constraints.

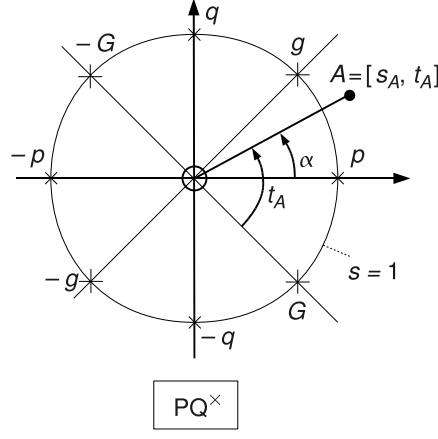
In order to select the system that could possibly pose as a natural, building-block type primitive for modeling a generalized nilpotence, we now argue that a preferred system requires a high degree of symmetry and a nondegenerate modulus along the axes generated by its basis elements. Figure 3, with  $r_{c1} = \cos(2\alpha)$  or  $r_{c2} = |\cos(2\alpha)|$ , offers the simplest realization of these requirements, as well as satisfying the constraints from section 2.2 above. Because  $r_{c2}$  is nonnegative for all  $\alpha$ , it traces a counterclockwise power orbit with increasing angle. While it is arguable that this property is more desirable as compared to differentiability at the coordinate origin, the power orbit  $r_{c2} = |\cos(2\alpha)|$  is now selected for further investigation.

### 3 PQ space

#### 3.1 $PQ^\times$ as multiplicative group with zero

One of the chosen constraints for generalization of nilpotence in this paper, is to require a multiplicative group similar to the complexes. The product  $A \cdot B$  of two numbers  $A = (x_A, y_A)$  and  $B = (x_B, y_B)$  in the complex plane is obtained by multiplying their norms, and adding their angles, i.e.,  $\|A \cdot B\| = \|A\| \|B\|$  and  $\angle(AB) = \angle(A) + \angle(B)$ . The square root of the norm function  $\|A\| = \sqrt{x_A^2 + y_A^2}$  becomes the Euclidean distance from the origin  $r_A := \sqrt{x_A^2 + y_A^2}$ , and corresponds to the radius  $r_A$  of circles of isonorm.

Figure 5. The multiplicative set  $PQ^\times$  is described by modulus  $s_A$  and angle  $t_A$ .



The law for composition algebras will now be generalized, by separating the radius function  $r_A$  from a generalized modulus function  $|A| := f(x_A, y_A)$ , and requiring a product-modulus law (in contrast to using norms):

$$|A \cdot B| = |A| |B|. \quad (21)$$

Notation for a point  $A$  uses modulus  $s_A := |A|$ , angle  $t_A$ , and square brackets:

$$A := [s_A, t_A]. \quad (22)$$

Multiplication “ $\times$ ” between two points is then defined as a multiplicative group with zero:

**Definition 1** The multiplicative set  $PQ^\times := \{[s, t]; s \geq 0; s, t \in \mathbb{R}\}$  contains a commutative multiplicative group for  $s > 0$  defined by:

$$\times : PQ^\times \otimes PQ^\times \rightarrow PQ^\times, \quad \text{where } \otimes \text{ means "set-cross"}, \quad (23)$$

$$A \times B = [s_{A \times B}, t_{A \times B}], \quad \text{with:} \quad (24)$$

$$s_{A \times B} = |A \times B| := |A| |B| = s_A s_B, \quad (25)$$

$$t_{A \times B} = \angle(A \times B) := \angle(A) + \angle(B) = t_A + t_B. \quad (26)$$

In the case  $s = 0$  there is for any  $Z := [0, t_Z] \in PQ^\times$ :

$$s_{Z \times A} = s_{Z \times C} = 0. \quad (27)$$

Therefore,  $PQ^\times$  will be called a “multiplicative group with zero”, where “zero”  $[0]$  is equivalent to:

$$[0] \equiv \{[0, t], t \in \mathbb{R}\}, \quad (28)$$

$$[0] \times A = [0] = A \times [0] \quad \text{for any } A \in PQ^\times. \quad (29)$$

Figure 5 illustrates the geometry of  $PQ^\times$ . The unit circle is the locus of unit modulus  $s = 1$ , and the relation between coordinate angle  $\alpha_A$  and point angle  $t_A$  is a simple offset:

$$t_A := \alpha_A + \frac{\pi}{4}. \quad (30)$$

$PQ^\times$  is isomorphic to the multiplicative group with zero of the complexes  $\mathbb{C}^\times$  under the mapping:

$$* : \mathbb{C}^\times \rightarrow PQ^\times, \quad \text{where:} \quad (31)$$

$$\sqrt{\|A\|} \rightarrow s_A, \quad (32)$$

$$\alpha_A \rightarrow t_A - \frac{\pi}{4}. \quad (33)$$

This isomorphism maps the complex basis  $\{1, i\}$  to points in  $PQ^\times$  as:

$$\{1, i, -1, -i\} \leftrightarrow \{G, g, -G, -g\}, \quad (34)$$

$$\{\sqrt{i}, \sqrt{i^3}, \sqrt{i^5}, \sqrt{i^7}\} \leftrightarrow \{p, q, -p, -q\}. \quad (35)$$

Specifically, the point  $G$  is the image of 1 from the complexes, and marks the multiplicative identity element in  $\text{PQ}^\times$ :

$$1_{\mathbb{C}}^* = [1, 0]_{\mathbb{C}}^* = [1, 0]_{\text{PQ}^\times} = \text{id}^\times = G. \quad (36)$$

### 3.2 The vector space $\text{PQ}^+$

Addition “+” will be modeled as a vector space over the reals in the Euclidean  $\{p, q\}$  plane, and termed  $\text{PQ}^+$ . Notation for a point  $A$  uses the real coefficients  $x_A$  and  $y_A$  for the  $p$  and  $q$  axes respectively, and round brackets:

$$A := (x_A, y_A). \quad (37)$$

Its distance from the origin is the Euclidean  $r_A = \sqrt{x_A^2 + y_A^2}$ , and the coordinate angle between the distance vector and the  $p$  axis is labeled  $\alpha_A$ .

Addition “+” is then explicitly defined as:

**Definition 2**  $\text{PQ}^+$  is a vector space over the reals, and consists of the set of vectors  $\{(x, y) := xp + yq; x, y \in \mathbb{R}\}$  with Cartesian coordinate addition and scalar multiplication:

$$A = (x_A, y_A) = (r_A \cos \alpha_A, r_A \sin \alpha_A) = r_A \left( \cos \left( t_A - \frac{\pi}{4} \right), \sin \left( t_A - \frac{\pi}{4} \right) \right), \quad (38)$$

$$B = (x_B, y_B) = (r_B \cos \alpha_B, r_B \sin \alpha_B) = r_B \left( \cos \left( t_B - \frac{\pi}{4} \right), \sin \left( t_B - \frac{\pi}{4} \right) \right), \quad (39)$$

$$+ : \text{PQ}^+ \otimes \text{PQ}^+ \rightarrow \text{PQ}^+, \quad (40)$$

$$A + B := (x_A + x_B, y_A + y_B), \quad (41)$$

$$\lambda A = (\lambda x_A, \lambda y_A) \quad \text{with } \lambda \in \mathbb{R}. \quad (42)$$

The  $\text{PQ}^+$  group is trivially isomorphic to the additive group of the complexes  $\mathbb{C}^+$  under the mapping:

$$\# : \mathbb{C}^+ \rightarrow \text{PQ}^+, \text{ where:} \quad (43)$$

$$x \rightarrow xp, \quad (44)$$

$$yi \rightarrow yq. \quad (45)$$

### 3.3 Defining PQ space similar to complex numbers $\mathbb{C}$

The complexes can be recovered from multiplication in  $\mathbb{C}^\times$  and vector addition in  $\mathbb{C}^+$  in the trivial case:

**Definition 3** Let  $\mathbb{C}^+$  be a vector space, and  $\mathbb{C}^\times$  a multiplicative group with zero, then “complex numbers”  $\mathbb{C} := \mathbb{C}^+ \cup \mathbb{C}^\times$  are the set and algebra where radius  $r$  and modulus  $s$  of a point  $V$  are the same, i.e.:

$$s_{\mathbb{C}} := r = \sqrt{x^2 + y^2}. \quad (46)$$

The distinction between modulus and radius is now used to formally define a number system that generalizes nilpotence as desired.

**Definition 4** Let  $\text{PQ}^+$  be a vector space,  $\text{PQ}^\times$  a multiplicative group with zero; then “PQ space” is the set

$$\text{PQ} := \text{PQ}^+ \cup \text{PQ}^\times, \quad (47)$$

algebras

$$+ : \text{PQ}^+ \otimes \text{PQ}^+ \rightarrow \text{PQ}^+, \quad (48)$$

$$\times : \text{PQ}^\times \otimes \text{PQ}^\times \rightarrow \text{PQ}^\times, \quad (49)$$

and projective mapping

$$r : \text{PQ}^\times \rightarrow \text{PQ}^+ \quad (50)$$

such that for a general point  $V := [s, t; r]$  with radius  $r$ , angle  $t$ , and modulus  $s := |V|$ , the following relations hold:

Figure 6. Mappings between  $\mathbb{C} = \mathbb{C}^+ \cup \mathbb{C}^\times$  and  $\text{PQ} = \text{PQ}^+ \cup \text{PQ}^\times$ .

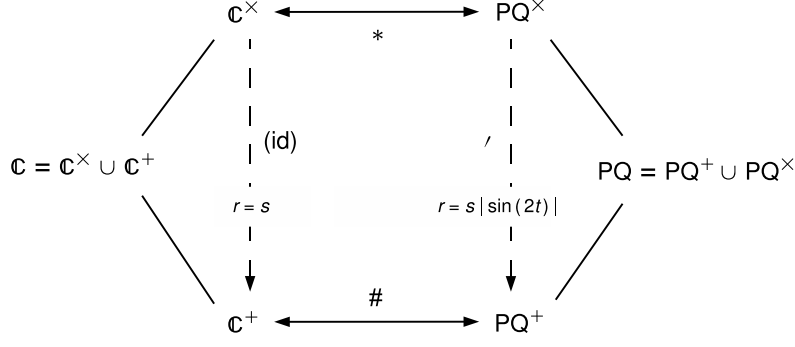
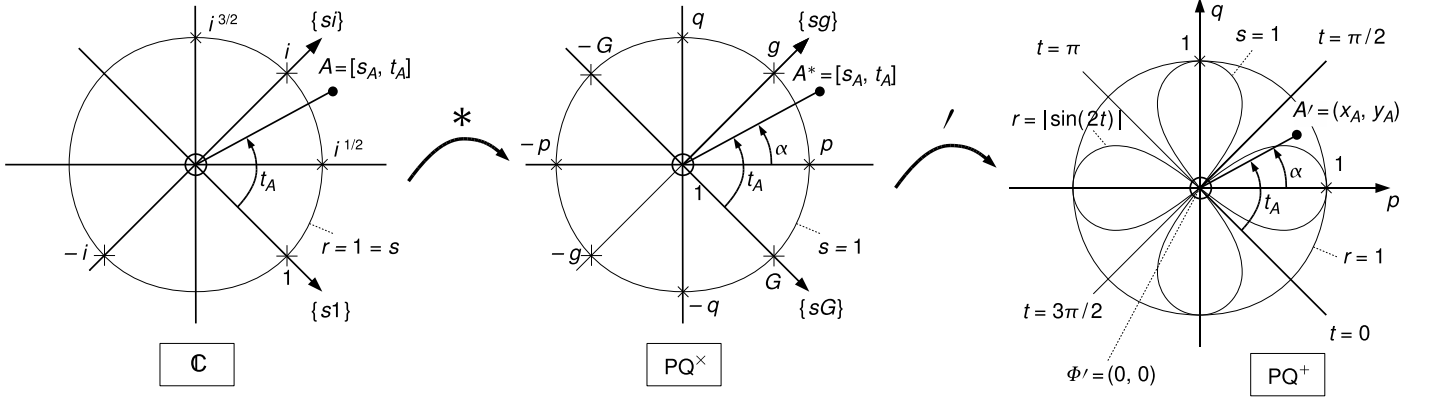


Figure 7. Isomorphic map  $*$ :  $\mathbb{C}^\times \rightarrow \text{PQ}^\times$ , and projective map  $\iota$ :  $\text{PQ}^\times \rightarrow \text{PQ}^+$ .



$$r := s |\sin(2t)|, \quad (51)$$

$$V' := (x, y) = r \left( \cos\left(t - \frac{\pi}{4}\right), \sin\left(t - \frac{\pi}{4}\right) \right) \text{ where } r := \sqrt{x^2 + y^2}. \quad (52)$$

This definition implies:

$$|V| = s = \frac{r^3}{|x^2 - y^2|} = \sqrt{x^2 + y^2} \left| \frac{x^2 + y^2}{x^2 - y^2} \right| = r \left| \frac{x^2 + y^2}{x^2 - y^2} \right| \text{ for } x \neq \pm y. \quad (53)$$

Figure 6 gives a schematical view over the different mappings that were introduced, to highlight the radius-modulus relation where PQ space differs from the complexes. Figure 7 illustrates the isomorphic map  $*$ :  $\mathbb{C}^\times \rightarrow \text{PQ}^\times$  that maps the circle of unit norm in the complexes into the circle of unit modulus in  $\text{PQ}^\times$ , and then the projective map  $\iota$ :  $\text{PQ}^\times \rightarrow \text{PQ}^+$  which maps the unit modulus from  $\text{PQ}^\times$  into a shape similar to a four-leaved clover in  $\text{PQ}^+$ . Table 2 gives the integral powers of  $p$  and  $q$  which are cyclic with a period of eight.

In PQ space, all multiplication occurs within  $\text{PQ}^\times$ ; all addition occurs within the  $\text{PQ}^+$  vector space; and scalar multiplication can exist on all of PQ:  $(\lambda V)' = \lambda (V')$ . Neither  $\text{PQ}^\times$  nor  $\text{PQ}^+$  form a preferred representation of PQ space. The vector space representation from figure 7 is chosen merely due to familiarity with the concept of vector addition. In total, PQ space consists of: the projective relation  $\iota$  between multiplicative set  $\text{PQ}^\times$  and vector space  $\text{PQ}^+$ , their union set and algebra under both  $\times$  and  $+$  operations.

### 3.4 The diagonal set $X_0$ , zero-center $\Phi$ , directed zeroes, and generalization of nilpotence

The map  $\iota$ :  $\text{PQ}^\times \rightarrow \text{PQ}^+$  projects all even powers of  $p$  and  $q$  into the coordinate origin of  $\text{PQ}^+$ , i.e.,  $(p^{2n})' = (q^{2n})' = (0, 0)$ . This makes the inverse map

$$-\iota: \text{PQ}^+ \rightarrow \text{PQ}^\times \quad (54)$$

Table 2

The eight integral powers of  $p$  and  $q$ .

$$\begin{aligned}
p^1 &= [1, \frac{\pi}{4}; 1] = p, & q^1 &= [1, \frac{3\pi}{4}; 1] = q, \\
p^2 &= [1, \frac{\pi}{2}; 0] = g, & q^2 &= [1, \frac{3\pi}{2}; 0] = p^6 = -g, \\
p^3 &= [1, \frac{3\pi}{4}; 1] = q^1, & q^3 &= [1, \frac{9\pi}{4}; 1] \equiv [1, \frac{\pi}{4}; 1] = p, \\
p^4 &= [1, \pi; 0] = -G, & q^4 &= [1, \pi; 0] = p^4 = -G, \\
p^5 &= [1, \frac{5\pi}{4}; 1] = -p, & q^5 &= [1, \frac{7\pi}{4}; 1] = -q, \\
p^6 &= [1, \frac{3\pi}{2}; 0] = q^2 = -g, & q^6 &= [1, \frac{5\pi}{2}; 0] \equiv [1, \frac{\pi}{2}; 0] = p^2 = g, \\
p^7 &= [1, \frac{7\pi}{4}; 1] = -q, & q^7 &= [1, \frac{5\pi}{4}; 1] = -p, \\
p^8 &= [1, 2\pi; 0] \equiv [1, 0; 0] = G = \text{id}^\times, & q^8 &= [1, 2\pi; 1] \equiv [1, 0; 1] = G = \text{id}^\times.
\end{aligned}$$

undefined on  $(0, 0) \in \text{PQ}^+$ , but also on the diagonals  $(x, y)$  where  $x = \pm y$ . In order to discuss the algebraic properties of PQ space, the diagonal set  $X_0$  and zero-center  $\Phi$  are now introduced together with a compact notation for implied maps under mixed  $\times$  and  $+$  operations.

**Definition 5** The “zero-center”  $\Phi$  is every point in PQ space with  $r = 0$ , i.e.:

$$V \in \Phi \Leftrightarrow V' = (0, 0). \quad (55)$$

It is the pre-image of  $(0, 0)$  under projection  $\iota$ . It includes the special case

$$[0]' = \{[0, t]', t \in \mathbb{R}\} = (0, 0). \quad (56)$$

The set  $\Phi$  can be written equivalently as:

$$\Phi = \{0V \text{ for any } V \in \text{PQ}\} \cup \{sZ \text{ where } Z \in \{G, g, -G, -g\}, 0 < s \in \mathbb{R}\} \quad (57)$$

$$= \{[0, x; 0] \text{ for any } x \in \mathbb{R}\} \cup \left\{ \left[ s, \frac{n\pi}{2}; 0 \right] \text{ for any } n \in \mathbb{Z}, 0 < s \in \mathbb{R} \right\}. \quad (58)$$

The zero-center is closed under  $\times$ , i.e.:

$$Z_1, Z_2 \in \Phi \Rightarrow Z_1 \times Z_2 \in \Phi. \quad (59)$$

Points  $[s, n\pi/2; 0] \in \Phi$  with nonzero modulus can be interpreted as “directed zeroes”: They contain an angle  $n\pi/2$  in  $\text{PQ}^\times$ , but map into  $(0, 0) \in \text{PQ}^+$ . The points  $\{G, g, -G, -g\}$  then form a “directed zero basis”. This is a key concept in PQ space. While the angle information is lost in the projective mapping  $\iota$ , one can envision future extensions of the system that contain a trivial kernel in a generalized map (as will be sketched in section 5.2).

**Definition 6** The “diagonal set”  $X_0$  are all points in  $\text{PQ}^+$  that cannot be mapped from  $\text{PQ}^\times$  using  $\iota$ , i.e.:

$$X_0 := \text{PQ}^+ \setminus \{V' \text{ for any } V \in \text{PQ}^\times\}. \quad (60)$$

The points in  $X_0$  are the coordinate diagonals in  $\text{PQ}^+$  without the origin, i.e.:

$$X_0 = \{(x, y) \text{ where } x = \pm y, y \neq 0\}. \quad (61)$$

A notation  $\widehat{\times}$  is now introduced for general multiplication of any two elements  $A, B \in \text{PQ} \setminus X_0$ , whether in  $\text{PQ}^+ \setminus X_0$  or  $\text{PQ}^\times$  (for values in  $X_0$  the  $\widehat{\times}$  is undefined):

$$\widehat{\times} : \text{PQ} \otimes \text{PQ} \rightarrow \text{PQ}^\times, \quad (62)$$

$$A \widehat{\times} B := \begin{cases} A \times B & \text{when } A, B \in \text{PQ}^\times, \\ A^{-1} \times B^{-1} & \text{when } A, B \in \text{PQ}^+ \setminus (X_0 \cup (0, 0)), \\ A \times B^{-1} & \text{when } A \in \text{PQ}^\times, B \in \text{PQ}^+ \setminus (X_0 \cup (0, 0)), \\ A^{-1} \times B & \text{when } A \in \text{PQ}^+ \setminus (X_0 \cup (0, 0)), B \in \text{PQ}^\times, \\ \text{undefined} & \text{when either } A \in X_0 \text{ or } B \in X_0. \end{cases} \quad (63)$$

A general product  $C$  will be abbreviated without explicitly writing the mappings:

$$C = A \times B. \quad (64)$$

Exponentiation of a point  $C$  to a real number  $a$  also implies that  $C \in \text{PQ}^\times$ , and will be written as  $C^a$ .

Similarly, notation  $\widehat{+}$  is now introduced for addition of any elements in  $\text{PQ}$ . For any  $A, B$ , whether in  $\text{PQ}^+$  or  $\text{PQ}^\times$ , their general sum  $A\widehat{+}B$  is:

$$\widehat{+} : \text{PQ} \otimes \text{PQ} \rightarrow \text{PQ}^+, \quad (65)$$

$$A\widehat{+}B := \begin{cases} A + B & \text{when } A, B \in \text{PQ}^+, \\ A' + B' & \text{when } A, B \in \text{PQ}^\times, \\ A + B' & \text{when } A \in \text{PQ}^+, B \in \text{PQ}^\times, \\ A' + B & \text{when } A \in \text{PQ}^\times, B \in \text{PQ}^+. \end{cases} \quad (66)$$

A general sum  $D$  will be abbreviated as:

$$D = A + B. \quad (67)$$

The abbreviated notation has no mathematical significance and is introduced only for convenience. It allows one to demonstrate the initial claim of a generalized nilpotence in one view: As compared to the complex numbers, for any  $A, B \in \text{PQ}$  the zero-center  $\Phi$  generalizes the additive identity zero,  $(0, 0)$ . So, for  $A, B \in \text{PQ}^\times$  we have:

$$A\widehat{+}B = A \iff A' + B' = A' \iff B' = (0, 0) \iff B \in \Phi. \quad (68)$$

Since

$$p^2, q^2 \in \Phi, \quad (69)$$

$$(p^2)' = (q^2)' = (0, 0), \quad (70)$$

$\text{PQ}$  space can be viewed as a doubly nilpotent number system in the two dimensional plane.

It is worth noting that although  $\text{PQ}^\times$  has a modulus  $|\cdot|$  this does not imply a norm  $\|\cdot\|$ , since any point  $V \in \Phi$  maps onto the null vector in  $\text{PQ}^+$ ,  $V' = (0, 0)$ , but the modulus may take any value  $|V| \geq 0$ . In contrast, a norm requires  $\|X\| = 0$  only when  $X = 0$ .

### 3.5 Notes on projection $\iota$ , homomorphism, zero-divisors, and additive identities

Briefly digressing, this section highlights apparent zero-divisors and additive identities when projection  $\iota$  would be defined on  $\text{PQ}^+$  to be the identity map of  $\text{PQ}^+$  elements:

$$(x, y)' := (x, y). \quad (71)$$

Then,  $\widehat{+}$  could be defined as:

$$A\widehat{+}B := A' + B', \quad (72)$$

$$[A\widehat{+}B]' = [A'\widehat{+}B'] = A' + B'. \quad (73)$$

Under such definitions,  $\iota$  would be an additive homomorphism from all  $\text{PQ}$  under  $\widehat{+}$  onto  $\text{PQ}^+$ .

However,  $\iota$  does not always behave as a multiplicative homomorphism. There exist  $A, B \in \text{PQ}$  with

$$A \times B = Z \in \Phi, \quad (74)$$

$$A, B \notin \Phi, \quad (75)$$

which implies:

$$(A \times B)' = (0, 0), \quad (76)$$

$$A' \neq (0, 0), B' \neq (0, 0). \quad (77)$$

Since  $\text{PQ}^\times \setminus [0]$  is a group, no zero-divisors can exist there. Both  $A'$  and  $B'$  merely *appear* as such in  $\text{PQ}^+$  because  $Z' = (0, 0)$ .

Similarly, when adding:

$$Z \in \Phi, A \in \text{PQ}^+, \quad (78)$$

$$A \widehat{+} Z = A' + Z' = A + (0, 0) = A, \quad (79)$$

then  $Z$  could be called an ‘‘additive identity in  $\text{PQ}$  under  $\widehat{+}$ ’’, even though this is generally not the case in the group  $\text{PQ}^+$  under  $+$ , where  $(0, 0)$  is the only additive identity. The zero-center  $\Phi$  then becomes the kernel of the extended projective map  $\iota$ .

### 3.6 Scalar multiplication, fractional powers, and division

Radius in the  $\text{PQ}^+$  plane,  $r = s|\sin(2t)|$ , is linear in  $\text{PQ}^\times$  modulus  $s$ , therefore an implied scalar multiplication by a factor  $0 < \lambda \in \mathbb{R}$  can be defined for any point  $V = [s, t]$ , denoted with radius as  $V = [s, t; r]$ , and  $V' = (x_V, y_V)$  by identifying:

$$\lambda V = \lambda [s, t; r] := [\lambda s, t; \lambda r] = [\lambda, 0; 0] \times \left[1, t; \frac{r}{\lambda}\right], \quad (80)$$

$$(\lambda V)' = \lambda (V') = (\lambda x_V, \lambda y_V). \quad (81)$$

The notation  $\lambda V$  is unambiguous regardless of whether  $V$  is a point in  $\text{PQ}^+$  or  $\text{PQ}^\times$ . For negative factors  $\lambda$  an angle shift  $t \rightarrow t + \pi$ ,  $\lambda \rightarrow |\lambda|$  is needed as the modulus  $s$  is positive definite. For  $\lambda = 0$  the product will be  $\lambda V = 0 = [0, t; 0]$  where  $t$  is indeterminate.

Fractional powers (and therewith, division) are defined in  $\text{PQ}^\times$  through linear interpolation of the angle  $t$ , just as in the complex numbers. For the basis elements  $p$  and  $q$  this is:

$$p^a := \left[1, \frac{a\pi}{4}; \left|\sin\left(\frac{a\pi}{2}\right)\right|\right], \quad a \in \mathbb{R}, \quad (82)$$

$$q^a := \left[1, \frac{3a\pi}{4}; \left|\sin\left(\frac{3a\pi}{2}\right)\right|\right]. \quad (83)$$

In general, for any point  $V$  with unit modulus and angle  $t$ , its real powers are:

$$V^a = [1, at; |\sin(at)|]. \quad (84)$$

For a general point  $V = [s, t; r]$  with  $s > 0$ ,  $V = s\widehat{V}$ , this becomes:

$$V^a := (s\widehat{V})^a \quad (85)$$

$$= s^a \widehat{V}^a = s^a [1, at; |\sin(2at)|] \quad (86)$$

$$= [s^a, at; s^a |\sin(2at)|]. \quad (87)$$

With  $\widehat{V} = p^b$  for  $b = \frac{4t}{\pi}$  this can also be written as:

$$V^a = (sp^b)^a = s^a p^{ba}. \quad (88)$$

In other words, exponentiation of a general point,  $V^a$  with  $a \in \mathbb{R}$ , scales the angle  $t \rightarrow at$  and exponentiates the modulus  $s \rightarrow s^a$ .

The inverse,  $V^{-1}$ , is then:

$$V^{-1} := \left[\frac{1}{s}, -t; \frac{|\sin(-2t)|}{s}\right] = \left[\frac{1}{s}, -t; \frac{r}{s^2}\right] = s^{-1} p^{-b} \quad \text{for } s > 0, \quad (89)$$

$$V \times V^{-1} = \left[s \frac{1}{s}, t - t; |\sin(2t - 2t)|\right] = [1, 0; 0] = G. \quad (90)$$

### 3.7 Nondistributivity

Multiplication in  $\text{PQ}^\times$  generally does not distribute over addition in  $\text{PQ}^+$ . A simple example is a point  $A$  which can be uniquely mapped into the  $\{p, q\}$  plane, and expressed in Cartesian coordinates  $A' = (x_A, y_A)$ :

$$A := [s_A, t_A; r_A] \mapsto A' = (x_A, y_A) = x_A p + y_A q. \quad (91)$$

Multiplying both representations of  $A$  on either side with itself, under false assumptions ( $\stackrel{??}{=}$ ) of distributivity, yields:

$$A \times A = [s_A^2, 2t_A; s_A^2 |\sin(4t_A)|] \neq (x_A p + y_A q) \times (x_A p + y_A q) \quad (92)$$

$$\stackrel{??}{=} x_A^2 p^2 + 2x_A y_A (p \times q) + y_A^2 q^2 \quad (93)$$

$$\stackrel{??}{=} (x_A^2 - y_A^2) g - 2x_A y_A G \quad (94)$$

$$\xrightarrow{\prime} (0, 0). \quad (95)$$

Whereas the left side generally yields a nonzero radius  $r_{A \times A} = s_A^2 |\sin(4t_A)|$ , the radius function of both terms on the right side is always zero. Multiplication in  $\text{PQ}^\times$  cannot distribute over the homomorphic map  $\prime$  and addition in  $\text{PQ}^+$  in general. This is somewhat expected since  $\prime$  is a nonlinear map between points in the two dimensional plane,  $r = s |\sin 2t|$ .

Equation (95) might suggest that distributivity exists within  $\Phi$ , since  $\Phi' = (0, 0)$ . However, for  $Z_1, Z_2, Z_3 \in \Phi$  the expression

$$Z_1 \times (Z_2 + Z_3) \stackrel{??}{=} Z_1 \times ((0, 0)) \quad (96)$$

is invalid, as  $(0, 0)^{-\prime}$  is undefined.

### 3.8 Polynomials and exponential function

A general polynomial over a point  $V = [s, t; r]$ ,  $V \in \text{PQ}^\times$ , with real coefficients  $z_n$ ,

$$Z(V) := \sum_{n=1}^{\infty} z_n V^n \quad (97)$$

$$= z_1(V) + z_2(V \times V) + z_3(V \times V \times V) + \dots + z_n(V \overset{n \text{ times}}{\times} \dots \times V) + \dots, \quad (98)$$

is a combination of addition  $+$  and multiplication  $\times$  in  $\text{PQ}$ . The map  $\prime : \text{PQ}^\times \rightarrow \text{PQ}^+$  is implied when summing (using the shorthand notation defined in section 3.4).

As example, one can define an exponential function  $\exp V$  through its Taylor polynomial:

$$\exp V := \sum_{n=0}^{\infty} \frac{1}{n!} V^n \quad (99)$$

$$= [1, 0; 0] + [s, t; s |\sin(2t)|] + \frac{1}{2} [s^2, 2t; s^2 |\sin(4t)|] + \dots + \frac{1}{n!} [s^n, nt; s^n |\sin(2nt)|] + \dots \quad (100)$$

$$\xrightarrow{\prime} (0, 0) + s |\sin(2t)| \left( \cos\left(t - \frac{\pi}{4}\right), \sin\left(t - \frac{\pi}{4}\right) \right) + \dots + \frac{s^n}{n!} |\sin(2nt)| \left( \cos\left(nt - \frac{\pi}{4}\right), \sin\left(nt - \frac{\pi}{4}\right) \right) + \dots \quad (101)$$

For the special case

$$p = \left[ 1; \frac{\pi}{4}; 1 \right], \quad (102)$$

we have  $|\sin(2nt)| \in \{0, 1\}$ , and the expression  $\exp(sp)$  with  $s$  real reduces to:

$$\exp(sp) = \exp\left(\left[ s; \frac{\pi}{4}; s \right]\right) \xrightarrow{\prime} (0, 0) + s(1, 0) + (0, 0) + \frac{s^3}{3!}(0, 1) + (0, 0) + \frac{s^5}{5!}(-1, 0) + \dots \quad (103)$$

$$= \left( s - \frac{s^5}{5!} + \frac{s^9}{9!} - \dots, \frac{s^3}{3!} - \frac{s^7}{7!} + \frac{s^{11}}{11!} - \dots \right) \quad (104)$$

$$= \left( \sum_{n=0}^{\infty} (-1)^n \frac{s^{(4n+1)}}{(4n+1)!}, \sum_{n=0}^{\infty} (-1)^n \frac{s^{(4n+3)}}{(4n+3)!} \right). \quad (105)$$

The last line can be expressed in closed form by substituting  $x := s\sqrt{i} \in \mathbb{C}$  (where  $i = \sqrt{-1}$ ) into:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{(2n+1)}}{(2n+1)!}, \quad (106)$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots = \sum_{n=0}^{\infty} \frac{x^{(2n+1)}}{(2n+1)!}, \quad (107)$$

Then the closed form identities for relation (105) are:

$$\sum_{n=0}^{\infty} (-1)^n \frac{s^{(4n+1)}}{(4n+1)!} = \frac{1}{2i^{\frac{1}{2}}} \left( \sin \left( i^{\frac{1}{2}} s \right) + \sinh \left( i^{\frac{1}{2}} s \right) \right), \quad (108)$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{s^{(4n+3)}}{(4n+3)!} = \frac{i^{\frac{1}{2}}}{2} \left( \sin \left( i^{\frac{1}{2}} s \right) - \sinh \left( i^{\frac{1}{2}} s \right) \right). \quad (109)$$

These functions of complex numbers are used merely for convenience, as they allow us to express (105) with known functions; the expressions, however, will always evaluate to a real number.

### 3.9 Coordinate multiplication

At times it may be useful to evaluate PQ space multiplication in the  $\text{PQ}^+$  plane only. Formally, this would take two points  $A, B$  in the  $\{p, q\}$  plane of  $\text{PQ}^+$ , and find the points  $A^{-'} = [s_A, t_A; r_A]$  and  $B^{-'} = [s_B, t_B; r_B]$  in  $\text{PQ}^\times$  (if existing). These are multiplied to  $C = (A^{-'}) \times (B^{-'}) = [s_A s_B, t_A + t_B; s_A s_B |\sin(2(t_A + t_B))|]$ , and projected back into  $\text{PQ}^+$ ,  $C \mapsto C'$ . Coordinate multiplication is not possible for points in the diagonal set  $X_0$  or the origin  $(0, 0)$ , since the projective mapping  $-'$  is undefined there.

Using normalized points,

$$\hat{A} := \frac{1}{s_A} A = (x_A, y_A), \quad \hat{B} := \frac{1}{s_B} B = (x_B, y_B), \quad |\hat{A}| = |\hat{B}| = 1, \quad (110)$$

and:

$$X := x_A x_B - y_A y_B, \quad Y := x_A y_B + y_A x_B, \quad F := \frac{\sqrt{2} |XY|}{(X^2 + Y^2)^{\frac{3}{2}}}, \quad (111)$$

the result of such coordinate multiplication is:

$$A \times B = s_A s_B \hat{A} \times \hat{B} = s_A s_B [F(X - Y, X + Y)]. \quad (112)$$

As usual, the shorthand notation implies:

$$A \times B \equiv ((A^{-'}) \times (B^{-'}))'. \quad (113)$$

For derivation of this result, see appendix B. Next to scaling factor  $s_A s_B F$ , the vector part  $(X - Y, X + Y)$  in equation (112) reflects the isomorphism  $*$ :  $\mathbb{C}^\times \rightarrow \text{PQ}^\times$  that rotates the complex product  $(X, Y)$  by 45 degrees.

## 4 A comparison with “ $p$ and $q$ numbers” after C. Musès

Over the years, Charles A. Musès proposed “ $p$  and  $q$  numbers” as part of his hypernumbers concept, to address generalization of nilpotence, similar to the aim of this paper. While there were changes to his proposal over the years, the two requirements remained:  $p^2 = 0$  and  $|p| = 1$ . This section references and traces this concept over time, and then distinguishes it from PQ space in this paper.

Musès mentions in [2] a system

$$" |p| = 1, \text{ and } p^{\pm 2n} = 0, n = 1, 2, 3, \dots " \quad (114)$$

He suggests relations “ $1/p, (\equiv \bar{p}) \neq p^{-1} = -p$ ”, “ $p^2 = 0 = p^0$ ”, and

$$" e^{\pm \theta p} = |\sin \theta| (\cos \theta \pm p \sin \theta). " \quad (115)$$

As for real powers of  $p$ , Musès writes: “ $[p]$  ... shows a double tangent circle as its field form.” This seems to suggest real powers  $r_{a3} = |\sin \alpha|$  from figure 2, or similar. He then suggests  $q$  with “ $q^4 = 0; q \neq q^2 \neq q^3 \neq 0; |q| = 1$ ” to be “the proper fourth root of zero”. No further algebraic clarification is given.

In [3], Musès suggests "... $p^2 = 0$ , although  $p \neq 0$ ,  $p$  lying along an axis perpendicular to any  $i_n$  or  $\varepsilon_n$  axis<sup>1</sup>, and also perpendicular to the axis of ordinary numbers, i.e. the  $r$ -axis. [...] we have:  $p^3 = q$ ;  $q^3 = p^5 = -p$ ;  $q^5 = -q = p^7$ ; etc. [...] the power orbit of  $\pm p$  is the quadrifolium given in Cartesian form by  $(x^2 + y^2)^3 = (x^2 - y^2)^2$ , where the  $x$  axis corresponds to the  $p$  axis and the  $y$  axis corresponds to the  $q$  axis". This suggests real powers with radius relation  $r_{c1} = \cos(2\alpha)$  or  $r_{c2} = |\cos(2\alpha)|$ , as in figure 1. While this is closer to PQ space developed in this paper, we comment that no further algebraic properties of such a space is given. In particular,  $p^2 = 0$  seems to imply  $p^2 p = 0p = 0$  and therefore  $p^3 \neq p^2 p$ ; though this can only be speculated in the absence of definition of multiplication.

In subsequent publications [4,5,6], Musès changed the radius function of the proposed real powers, to be as in figure 4. In [4] he writes: "...  $p$ , with a power orbit in the form of the curve given in polar coordinates by

$$r = \cos 2\theta (\cos \theta - \sin \theta), \quad (116)$$

the definition being  $p^2 = 0$ ,  $p \neq 0$  [...]. We have also  $p^3$  orthogonal to  $p$ , giving the new phenomenon of a twin hypernumber, i.e.,  $p^3 = q$  and  $q^3 = p$ . Then  $p^5 = p$ ; and  $p^{2n} = 0$ , where  $n = 0, \pm 1, \pm 2, \dots$ "

This is specified further in [5]: "Fig 2. The power orbit of the hypernumber  $p$  (that of  $-p$  is its reflection in the line along  $\theta = -\pi/4$ ). The Cartesian equation is  $r^2(x-y)^2(x+y) = (x^2+y^2)^2$  where for the simple unit  $p$ ,  $r = 1$ . The polar form  $\rho^4 = r^2(\cos \theta - \sin \theta)^2(\cos \theta + \sin \theta)$ . [...] though coordinates can be used for addition, powers must be used in multiplication." He then gives examples of vector space addition. Multiplication first expresses two points  $P_1$  and  $P_2$  as " $P_1 = r_1 p^{k_1}$  and  $P_2 = r_2 p^{k_2}$ ", with  $r_1, r_2, k_1, k_2 \in \mathbb{R}$ , and an example suggests that the general multiplication is  $P_1 P_2 = r_1 r_2 p^{k_1 + k_2}$ .

Musès further writes: "It must be remembered that because  $p$  is nilpotent ( $p^2 = 0$ ,  $p \neq 0$ ), its zeroth power cannot be 1; in fact  $p^0 = 0$ ." We comment that this approach is ambiguous, as for  $k_1 = 0$  and  $r_1 = 1$ , we have  $P_1 = 0$  (since  $p^0 = 0$ ) and therefore  $P_1 P_2 = 0 P_2 = 0$ , but also  $P_1 P_2 = r_2 p^{k_2} = P_2$ . This would require  $P_2 = 0$ , in contrast to the envisioned system. Musès adds: "Hence also  $p^{-1} \neq 1/p$ , and since  $(1/p)(1/p) = (1/p^2) = \infty$ , we see that  $1/p$  is panpotent, i.e., a root of infinity." The expression  $1/p$  is not defined further, and it is unclear how it can relate to division if  $1/p \neq p^{-1}$ .

## 5 Possible extensions

### 5.1 A dual multiplication $PQ^\circ$

In an earlier paper we introduced "W space" [7], where two dual morphisms ( $\times$  and  $\circ$ ) generalize multiplication and require a choice of space between  $+\mathbb{W} = \langle (x, y), +, \times \rangle$  and  $-\mathbb{W} = \langle (x, y), +, \circ \rangle$ .

Similarly, one could now envision a second multiplicative set,  $PQ^\circ$ , such that  $-G = [1, \pi; 0]$  is its identity element, and angles  $t$  are measured clockwise from an angle 45 degrees from the  $q$ -axis (instead of anti-clockwise from an angle -45 degrees from the  $p$ -axis as in  $PQ^\times$ ). Such  $PQ^\circ$  multiplication would be somewhat a mirror of  $PQ^\times$  along the  $x = y$  diagonal in the  $\{p, q\}$  plane:

**Definition 7** *The commutative multiplicative set  $PQ^\circ := \{[s, t]; s \geq 0; s, t \in \mathbb{R}\}$  contains a multiplicative group with zero under  $\circ$  multiplication defined by:*

$$A \circ B \mapsto [s_{A \circ B}, t_{A \circ B}], \text{ with:} \quad (117)$$

$$s_{A \circ B} = |A \circ B| := |A||B| = s_A s_B, \quad (118)$$

$$t_{A \circ B} = \angle(A \circ B) := \angle(A) + \angle(B) = (\pi - t_A) + (\pi - t_B) \quad (119)$$

$$= -(t_A + t_B), [\text{mod } 2\pi]. \quad (120)$$

The special case  $s = 0$ , "zero", contains the elements of  $PQ^\circ$  that are not group elements,  $[0] \equiv \{[0, t], t \in \mathbb{R}\}$ .

From relation (120) it follows immediately that for two general points  $A, B \in PQ$  there is a relation:

$$A \times B = -(A \circ B) \quad \text{where } -1 \equiv -G = [1, \pi; 0], \quad (121)$$

$$A \times B \overset{\prime}{\mapsto} (x_C, y_C) \iff A \circ B \overset{\prime}{\mapsto} (-x_C, -y_C). \quad (122)$$

Just as was done for W space, one could then develop a PQ dual space that contains these two dual multiplications, introduce the notion of "representational equality", a "layered plane" interpretation, and other consequences discussed in [7].

<sup>1</sup> Here,  $i_n^2 = -1$  and  $\varepsilon_n^2 = 1$  are some basis elements.

## 5.2 Possible extensions of PQ space: Addition in the zero-center $\Phi$ , other radius functions

The constraints described for a generalization of nilpotence (section 2.2) can be satisfied in many ways. This section describes two examples how PQ space could be extended or varied: providing a trivial kernel in the projective map from a multiplicative group with zero to an additive group, and modifying the radius function.

As discussed in section 3.5, the zero-center  $\Phi$  is the kernel of the extended projective map  $\iota: \text{PQ} \rightarrow \text{PQ}^+$ . For any  $A \in \text{PQ}^\times$  and  $Z_1, Z_2 \in \Phi$ , this implies multiple identity elements:

$$Z'_1 = Z'_2 = (0, 0), \quad (123)$$

$$Z_1 + Z_2 \equiv Z'_1 + Z'_2 = (0, 0), \quad (124)$$

$$A + Z_1 \equiv A' + Z'_1 = A'. \quad (125)$$

This makes the map  $\iota$  not invertible for elements of  $\Phi$ , i.e.,  $(Z'_1)^{-\iota}$  is undefined. An extension of PQ space can be envisioned with generalized map from PQ into  $\text{PQ}^+$  that has a trivial kernel and is invertible. The challenge here would be not to lose the angle information when mapping a point from the zero-center into the generalized coordinate origin. The directed zero basis  $\{G, g, -G, -g\} \in \text{PQ}^\times$  from section 3.4 would need to become part of a generalization of the additive group  $\text{PQ}^+$ .

Another modification could be to the radius function for  $\iota$ , which was chosen to be  $r(t) = |\sin(2t)|$  for PQ space (figure 6). Depending on a future application of the system, one might choose different periodic radius functions  $r(t)$  with  $r(t_n) = 0$  for  $0 \leq t_n < 2\pi$ , and  $0 \leq n < N$  a whole number. This would include as special cases: the complex numbers with  $N = 0$ ,  $r(t) = 1$ , and PQ space with  $N = 4$ ,  $r(t) = |\sin(2t)|$ .

## 6 Summary, possible applications, and outlook

In this paper, we introduced PQ space as a doubly nilpotent number system in the two dimensional plane. We offered rich arithmetic including addition, subtraction, multiplication, division, and exponentiation. Limitations only applied to distinct, one dimensional subsets. Defining relations were introduced as similar as possible to the complex numbers, by requiring a modulus from the multiplicative set  $\text{PQ}^\times$  as generalized “length” function, and a nonlinear relation between this modulus and the radius used in the vector space  $\text{PQ}^+$ . The naturalness of choice for the radius-modulus-angle relation was defended in comparison with other possible choices, including “ $p$  and  $q$  numbers” after Charles A. Musès’ “hypernumbers” concept. Possible extensions were briefly sketched such as: introduction of a second multiplication group  $\text{PQ}^\circ \setminus [0]$  that is dual and isomorphic to  $\text{PQ}^\times \setminus [0]$ ; a speculated space that joins the vector space  $\text{PQ}^+$  with the kernel of the projective map  $\iota: \text{PQ} \rightarrow \text{PQ}^+$ ; and a variation of the radius function from  $\iota$ .

Possible applications for such a PQ space might include supersymmetry in physics, which uses expressions:

$$\{p, p\} := pp + pp = 0, \quad (126)$$

$$p \neq 0. \quad (127)$$

Rather than requiring  $p$  to be an operator, as is customary, it could now be speculated that PQ space may be useful for modeling certain relations algebraically:

$$p^2 = g \mapsto (0, 0), \quad q^2 = -g \mapsto (0, 0), \quad |p| = |q| = 1, \quad (128)$$

$$p, q \neq [0, 0; 0] \equiv "0". \quad (129)$$

Other uses of PQ space may include chaotic systems or catastrophe theory. As an example, effects from small variation around a critical point could be modeled by a small neighborhood around the coordinate origin in  $\text{PQ}^+$ : For any radius  $r > 0$  and arbitrary angle  $t$ , the range of possible moduli is  $\{s\} = [r, \infty)$ , since  $s = r/|\sin(2t)|$ . Multiplication then could yield a wide range of possible outcomes in the system under investigation. Social behavior, or even psychological behavior, often exhibits chaotic patterns, and may be of specific interest for modeling situations that seemingly arise “out of nowhere”.

As a visual treat, appendix A contains a fractal shape that results from using the Mandelbrot set algorithm,  $A_{n+1} = A_n^2 + C$ , for points  $A_n, C$  in the  $\{p, q\}$  plane. Of course, “beauty” is a rather vague argument to support naturalness of an algebra; but the butterfly-shaped fractal may be inspiring nonetheless.

The goal of this paper has been to establish, on sound mathematical footing, a doubly nilpotent space in a two dimensional plane, and to do this with as little as needed modifications as compared to the complexes. We hope that sufficient tools are now available to examine PQ space for potential application in nature.

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- [7] J. A. Shuster, J. Köplinger, Elliptic complex numbers with dual multiplication, *Appl. Math. Computation* **216** (2010) 3497-3514.
- [8] PQ space fractals of Mandelbrot type (bitmap samples, and C source code), <http://jenskoeplinger.com/PQ>.

## Appendix A: Fractal of Mandelbrot type in PQ space

One of the most widely known fractal shapes in the complex numbers is the Mandelbrot set, which contains all nondivergent points  $c$ , such that the infinite iteration

$$z_0 := 0, \tag{130}$$

$$z_{n+1} := z_n^2 + c, \tag{131}$$

is not divergent, i.e.,  $|z_n|$  finite for any  $n$ .

Similarly, this algorithm can now be applied to PQ space, using coordinate multiplication from section 3.9. The seed value is  $A_0 := C \in \text{PQ}^+$  where  $C = (x_c, y_c)$  corresponds to the location along  $p$ - and  $q$ -axis of the output bitmap. If  $(A_0)^{-'}$  is defined, then the value is projected into  $\text{PQ}^\times$ , multiplied with itself, and then projected back into  $\text{PQ}^+$ . The iteration then adds  $C$  again, and so on. If  $(A_n)^{-'}$  is undefined then the iteration is considered divergent:

$$A_0 := C, \tag{132}$$

$$A_{n+1} := \left( \left[ (A_n)^{-'} \right]^2 \right)' + C. \tag{133}$$

Figure 8 shows the outcome of this algorithm (source code and more samples at [8]), with nondivergent points  $C$  colored in black, taking on a shape that resembles a butterfly. The symmetry axis  $y = -x$  coincides with angles  $t \in \{0, \pi\}$ , corresponding to  $\{G, -G\}$  of the multiplicative group, and therefore reflects symmetry about the line of the multiplicative identity element,  $\text{id}^\times = G \in \text{PQ}^\times$ .

## Appendix B: Deriving coordinate multiplication

This appendix derives the result for coordinate multiplication from section 3.9. The procedure is as follows:

- Take two points in  $\text{PQ}^+$ , in the  $\{p, q\}$  plane:  $A := (x_A, y_A)$  and  $B := (x_B, y_B)$ .
- Find their equivalent points  $A^{-'} = [s_A, t_A; r_A]$  and  $B^{-'} = [s_B, t_B; r_B]$  in  $\text{PQ}^\times$ . This is only possible for  $x_A \neq \pm y_A$  and  $x_B \neq \pm y_B$ . For points in the diagonal set  $X_0$  (i.e.,  $x_A = \pm y_A$  or  $x_B = \pm y_B$ ), coordinate multiplication is undefined.
- Multiply  $A$  and  $B$  to  $C := A \times B \equiv (A^{-'}) \times (B^{-'}) = [s_A s_B, t_A + t_B; s_A s_B |\sin(2(t_A + t_B))|]$ .
- Finally, project  $C$  back into its equivalent coordinates in  $\text{PQ}^+$ , i.e.,  $C \overset{\cdot}{\mapsto} C' = (x_C, y_C)$ .

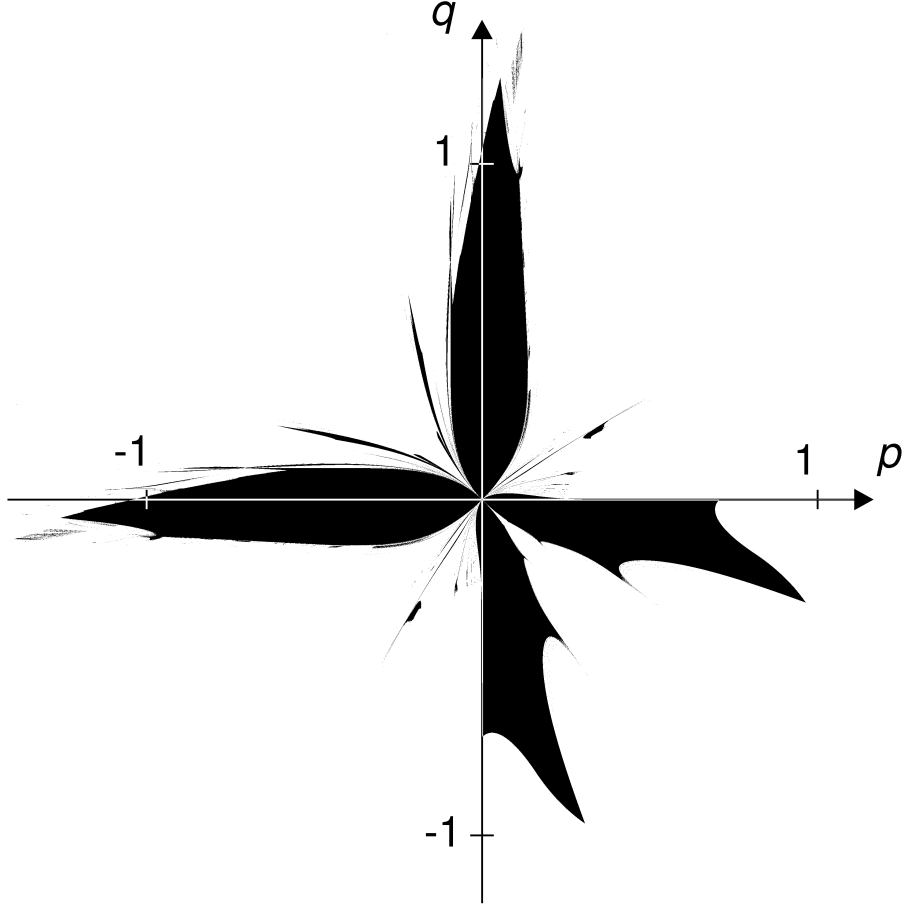
The following will be defined:

$$A := [s_A, t_A; r_A] = \left[ s_A, \alpha_A + \frac{\pi}{4}; r_A \right] \tag{134}$$

$$\hat{A} := \frac{1}{s_A} A \overset{\cdot}{\mapsto} (x_A, y_A), \quad \hat{B} := \frac{1}{s_B} B \overset{\cdot}{\mapsto} (x_B, y_B), \quad \hat{C} := \frac{1}{s_C} C \overset{\cdot}{\mapsto} (x_C, y_C), \quad |\hat{A}| = |\hat{B}| = |\hat{C}| = 1, \tag{135}$$

$$X := x_A x_B - y_A y_B, \quad Y := x_A y_B + y_A x_B \quad (X \neq Y). \tag{136}$$

Figure 8. The set of nondivergent points  $C$  from Mandelbrot algorithm  $A_{n+1} = A_n^2 + C$  in PQ space, with  $A_0 = C$ .



Using the identities

$$\tan(t_A + t_B) = \frac{\tan t_A + \tan t_B}{1 - \tan t_A \tan t_B} \quad (\text{where } 1 - \tan t_A \tan t_B \neq 0), \quad (137)$$

$$\tan \alpha_A = \frac{y_A}{x_A} \quad (\text{where } x_A \neq 0), \quad (138)$$

$$\tan t_A = \tan\left(\alpha_A + \frac{\pi}{4}\right) = \frac{x_A + y_A}{x_A - y_A} \quad (\text{where } x_A - y_A \neq 0), \quad (139)$$

and similarly for  $\tan \alpha_B$ ,  $\tan \alpha_C$ ,  $\tan t_B$ , and  $\tan t_C$ , we have:

$$\frac{x_C + y_C}{x_C - y_C} = \tan t_C = \tan(t_A + t_B) = \frac{\tan t_A + \tan t_B}{1 - \tan t_A \tan t_B} = \frac{\frac{x_A + y_A}{x_A - y_A} + \frac{x_B + y_B}{x_B - y_B}}{1 - \frac{(x_A + y_A)(x_B + y_B)}{(x_A - y_A)(x_B - y_B)}} \quad (140)$$

$$= \frac{(x_A + y_A)(x_B - y_B) + (x_A - y_A)(x_B + y_B)}{(x_A - y_A)(x_B - y_B) - (x_A + y_A)(x_B + y_B)} = \frac{x_A x_B - y_A y_B}{-x_A y_B - y_A x_B} = -\frac{X}{Y}. \quad (141)$$

Therefore,

$$y_C = x_C \frac{X+Y}{X-Y} \quad (\text{where } X-Y \neq 0), \quad (142)$$

$$\hat{A} \times \hat{B} \mapsto (x_C, y_C) = \left( x_C, x_C \frac{X+Y}{X-Y} \right) = \frac{x_C}{X-Y} (X-Y, X+Y) = F (X-Y, X+Y). \quad (143)$$

The factor  $F$  normalizes  $s_C = |\hat{A} \times \hat{B}| = 1$  as required, so:

$$1 = |\hat{A} \times \hat{B}| = \frac{(x_C^2 + y_C^2)^{\frac{3}{2}}}{|x_C^2 - y_C^2|} \quad (144)$$

$$= F \frac{\left( (X-Y)^2 + (X+Y)^2 \right)^{\frac{3}{2}}}{\left| (X-Y)^2 - (X+Y)^2 \right|} = F \frac{(2X^2 + 2Y^2)^{\frac{3}{2}}}{|-4XY|}, \quad (145)$$

$$\implies F = \frac{\sqrt{2}|XY|}{(X^2 + Y^2)^{\frac{3}{2}}}. \quad (146)$$

This yields the result:

$$A \times B = s_{ASB} \hat{A} \times \hat{B} = C \stackrel{!}{\mapsto} C' = s_{ASB} [F (X-Y, X+Y)]. \quad (147)$$