Gravity and electromagnetism on conic sedenions

Jens Köplinger

406 Aberdeen Ter, Greensboro, NC 27403, USA

Abstract

Conic sedenions from C. Musès' hypernumbers program are able to express the Dirac equation in physics through their hyperbolic subalgebra, together with a counterpart on circular geometry that has earlier been proposed for description of gravity. An electromagnetic field will now be added to this formulation and shown to be equivalent to current description in physics. With use of an invariant hypernumber modulus condition, a description of quantum gravity with field will be derived. The resulting geometry reduces in very good approximation to relations expressible through customary tensor algebra. However, deviations are apparent at extreme energy levels, as shortly after the Big Bang, that require genuine conic sedenion arithmetic for their correct description. This is offered as method for exploration into bound quantum states, which are not directly observable in the experiment at this time. Extendibility of the invariant modulus condition to higher hypernumber levels promises mathematical flexibility beyond gravity and electromagnetism.

Key words: hypernumbers, quantum gravity, electromagnetism, conic complex numbers, countercomplex numbers, sedenions, octonions, physics on hyperbolic and circular geometry

1 Introduction

In order to qualify and sufficiently support a quantum theory of gravitation on genuine conic sedenion arithmetic (from C. Musès' hypernumbers program), it was first shown that the Dirac equation in physics is expressible on hyperbolic octonion arithmetic [1]. Rotation in the $(1, i_0)$ plane yields a counterpart on circular (Euclidean) geometry, which exhibits certain primitive properties of quantum gravity [2]. For large-body (non-quantum) physics, an alignment program was developed from an invariant modulus condition [3], and shown to be consistent with the General Relativity formalism for gravity.

This paper will conclude definition of the computational framework for a proposed quantum theory of gravitation and electromagnetism on hypernumbers, by supplying a physical force field to the formulation. An electromagnetic field will be added to the hyperbolic subalgebra and shown to be equivalent to current description in physics, supporting a conjecture by analogy for the circular subalgebra to describe quantum gravitational interaction.

Certain mixing effects become apparent at extreme energies, which cannot be separated into the individual constituent forces of gravity and electromagnetism anymore. This makes conic sedenion arithmetic a needed tool for further investigation into such effects.

Referring to the power orbit concept, it will be remarked that hypernumbers offer additional mathematical versatility to satisfy a generalized invariant hypernumber modulus theorem, beyond the description of gravity and electromagnetism herein.

2 Electromagnetism

The Dirac equation with electromagnetic field is a fundamental building block in describing dynamic interaction of spin 1/2 particles (like electrons or protons). In a simple form it can be written as

$$[\gamma (\hat{p} - eA) - m] \Psi = 0 \tag{1}$$

(from [4] equation 32.1) and contains a operator $\hat{p}$ and certain implicit summations. In order to map this relation to conic sedenion arithmetic as in [1], it will now be written explicitely [1] in terms of space-time derivatives ∂_μ.

Email address: jens@prisage.com (Jens Köplinger).

[1] Please note that physical constants c, h, and G are set to 1, and all summations will be written using lower indices only. A metric tensor will be specified if present.

The terms γ, $\hat{p}$, A, and Ψ in (1) are summed over three indices $\mu, \nu, \sigma \in \{0,1,2,3\}$, with implicit use of a metric tensor $\eta_{\mu\nu}$ (describing Minkowski metric; $\eta_{\mu\nu} = 0$ for $\mu \neq \nu$, $\eta_{00} = 1$, and $\eta_{\mathrm{ii}} = -1$ otherwise), and result in four equations now indexed with $\rho \in \{0,1,2,3\}$. The γ are constant, anti-commuting 4×4 matrices $[\gamma_\mu]_{\rho\sigma}$ on complex numbers, $\hat{p}$ are the quantum mechanical operators for energy $\hat{p}_0 \equiv i\partial_0$ and momentum $\hat{p}_{\mathrm{i}} \equiv -i\partial_{\mathrm{i}}$ ($\mathrm{i} \in \{1,2,3\}$), e is the electric charge of the spin 1/2 particle, m is its mass, and A_ν the electromagnetic field acting upon it. Both A_ν and Ψ_σ are a function of time $t \equiv x_0$ and space $\vec{x} \equiv \{x_1, x_2, x_3\}$.

With $A^*_\mu := \sum_{\nu=0}^{3} \eta_{\mu\nu} A_\nu$ one can then write (1) in the following form:

$$\sum_{\sigma=0}^{3} \left(\sum_{\mu,\nu=0}^{3} [\gamma_\mu]_{\rho\sigma} \eta_{\mu\nu} (\hat{p}_\nu - eA_\nu) - \delta_{\rho\sigma} m \right) \psi_\sigma = 0 \tag{2}$$

$$\sum_{\sigma=0}^{3} \left(\sum_{\mu=0}^{3} \left(i [\gamma_\mu]_{\rho\sigma} \partial_\mu - [\gamma_\mu]_{\rho\sigma} eA^*_\mu \right) - \delta_{\rho\sigma} m \right) \psi_\sigma = 0 \tag{3}$$

Interpreting the summation terms $i [\gamma_\mu]_{\rho\sigma} \partial_\mu - [\gamma_\mu]_{\rho\sigma} eA^*_\mu$ as coefficients ∂_μ and eA^*_μ to bases $i\gamma_\mu$ and γ_μ respectively, one finds the following algebraic relations:

(1) All 8 bases $i\gamma_\mu$ and γ_μ, as well as, the term $-m$ (to basis "1"$\equiv \delta_{\rho\sigma}$) are linear independent.
(2) Since all γ_μ anti-commute ($\gamma_\mu\gamma_\nu = -\gamma_\nu\gamma_\mu$ for $\mu \neq \nu$), the bases to the ∂_μ factors anti-commute, as well as the bases to the eA^*_μ.
(3) The eA^*_μ bases anti-commute with bases to ∂_ν factors for $\mu \neq \nu$, and commute for $\mu = \nu$.

Conic sedenion arithmetic to basis elements $b_{\mathrm{con16}} \in \{1, i_1, \ldots, i_7, i_0, \varepsilon_1, \ldots, \varepsilon_7\}$ and the definitions

$$\nabla^{\mathrm{EM}}_{\mathrm{con16}} := (0, \partial_0, 0, 0, 0, eA_3, -eA_2, eA_1, \quad 0, eA_0, 0, 0, 0, -\partial_3, \partial_2, -\partial_1) \tag{4}$$

$$\Psi^{\mathrm{EM}}_{\mathrm{con16}} := \left(\psi^{\mathrm{r}}_0, \psi^{\mathrm{i}}_0, \psi^{\mathrm{r}}_1, \psi^{\mathrm{i}}_1, 0, 0, 0, 0, \quad 0, 0, 0, 0, \psi^{\mathrm{r}}_2, -\psi^{\mathrm{i}}_2, -\psi^{\mathrm{r}}_3, -\psi^{\mathrm{i}}_3\right) \tag{5}$$

with $m \equiv (m, 0, \ldots, 0)$ allow to express (3) as:

$$\left(\nabla^{\mathrm{EM}}_{\mathrm{con16}} - m\right) \Psi^{\mathrm{EM}}_{\mathrm{con16}} = 0 \tag{6}$$

Proof: For $A_\mu = 0$ relation (6) reduces to the correct form of the Dirac equation without field on hyperbolic octonion algebra [2]. Embedded in conic sedenions, demanding invariance under a transformation of the wave function

$$\Psi^{\mathrm{EM}}_{\mathrm{con16}} \rightarrow \Psi^{\mathrm{EM}}_{\mathrm{con16}} e^{i_0 \chi} \tag{7}$$

in direct analogy to its description in physics (e.g. [4] relation 32.3) with

$$\partial_\mu \chi := eA_\mu \tag{8}$$

yields:

$$i_1\partial_0 \rightarrow i_1 (\partial_0 - i_0 eA_0) = i_1\partial_0 + \varepsilon_1 eA_0 \tag{9}$$

$$-\varepsilon_5\partial_3 \rightarrow \varepsilon_5 (-\partial_3 + i_0 eA_3) = -\varepsilon_5\partial_3 + i_5 eA_3 \tag{10}$$

$$\varepsilon_6\partial_2 \rightarrow \varepsilon_6 (\partial_2 - i_0 eA_2) = \varepsilon_6\partial_2 - i_6 eA_2 \tag{11}$$

$$-\varepsilon_7\partial_1 \rightarrow \varepsilon_7 (-\partial_1 + i_0 eA_1) = -\varepsilon_7\partial_1 + i_7 eA_1 \tag{12}$$

The number bases of the A_μ terms correspond to (4), and the conic sedenion bases $\{i_1, -\varepsilon_5, \varepsilon_6, -\varepsilon_7\}$ and $\{\varepsilon_1, i_5, -i_6, i_7\}$ associated with the ∂_μ and A_μ terms respectively satisfy above algebraic relations of the $i\gamma_\mu$ and γ_μ bases used in physics (change of sign in $A^*_\mu = \pm A_\mu$ does not alter these relations). This concludes the proof.

[2] See [1] equation (5), but note that definition (4) there is incorrect and should be $\nabla_{\mathrm{hyp8}} := (-m, \partial_0, 0, 0, 0, -\partial_3, \partial_2, -\partial_1)$, as well as definition (3) should be $\Psi_{\mathrm{hyp8}} := \left(\psi^{\mathrm{r}}_0, \psi^{\mathrm{i}}_0, \psi^{\mathrm{r}}_1, \psi^{\mathrm{i}}_1, \psi^{\mathrm{r}}_2, -\psi^{\mathrm{i}}_2, -\psi^{\mathrm{r}}_3, -\psi^{\mathrm{i}}_3\right)$. The different definitions were the result of using a conic sedenion multiplication table which identified the classical octonion element "l" with sedenion element $-i_4$ instead of i_4, therefore not being consistent with the cited sources [5,6].

3 Gravity

In an identical procedure to electromagnetism above, the circular octonionic Dirac equation proposed for the description of gravity in [2] (relation 9 there) can be extended by a field eA_0 using the definitions

$$\nabla^{\rm Gr}_{\rm con16} := (0, \partial_0, 0, 0, 0, \partial_3, -\partial_2, \partial_1, \quad 0, eA_0, 0, 0, 0, eA_3, -eA_2, eA_1) \tag{13}$$

$$\Psi^{\rm Gr}_{\rm con16} := \left(\psi^{\rm r}_0, \psi^{\rm i}_0, \psi^{\rm r}_1, \psi^{\rm i}_1, -\psi^{\rm r}_2, \psi^{\rm i}_2, \psi^{\rm r}_3, \psi^{\rm i}_3, \quad 0, 0, 0, 0, 0, 0, 0, 0\right) \tag{14}$$

to:

$$\left(\nabla^{\rm Gr}_{\rm con16} - m\right) \Psi^{\rm Gr}_{\rm con16} = 0 \tag{15}$$

This is obtained by demanding invariance of $\Psi^{\rm Gr}_{\rm con16}$ under the same transformation as (7) above,

$$\Psi^{\rm Gr}_{\rm con16} \rightarrow \Psi^{\rm Gr}_{\rm con16}\, e^{i_0 \chi} \tag{16}$$

(with identical definition of $\partial_\mu \chi := eA_\mu$ using electromagnetic field A_μ and electric charge e of the particle).

Such an ansatz implies that every spin 1/2 particle with mass at rest m also has an associated electrical charge e. While this is true for most fundamental particles (electron, muon, tau, all quarks, and associated anti-particles), neutrinos are currently assumed to have a mass at rest but no electric charge. Since they only interact through gravitation and the weak force (with associated weak charge), a formalism that would describe both gravity and weak interaction could be more appropriate for these particles. They are therefore excluded from the current investigation.

In order to define the computational framework for a proposed quantum theory of gravitation and electromagnetism on hypernumbers, the invariant hypernumber modulus theorem from [3] will now be proposed to be valid for quantum gravity. This theorem was concluded to be sufficiently founded for large-body (non-quantum) gravitation through the *NatAliE* equations program, and rotation in the $(1, i_0)$ plane appears to be a plausible way to transform proven relations for electromagnetism on hyperbolic geometry into gravity on circular geometry.

With use of a real-number mixing angle α and the definitions

$$\nabla^{\rm Gr,EM}_{\rm Q1} := (0, \partial_0, 0, 0, 0, 0, 0, 0, \quad 0, eA_0, 0, 0, 0, 0, 0, 0) \tag{17}$$

$$\nabla^{\rm Gr,EM}_{\rm Q2} := (0, 0, 0, 0, 0, \partial_3, -\partial_2, \partial_1, \quad 0, 0, 0, 0, 0, eA_3, -eA_2, eA_1) \tag{18}$$

$$\nabla^{\rm Gr,EM} := \nabla^{\rm Gr,EM}_{\rm Q1} + \exp\left(i_0 \alpha\right) \nabla^{\rm Gr,EM}_{\rm Q2} \tag{19}$$

$$\Psi^{\rm Gr,EM}_{\rm Q1} := \left(\psi^{\rm r}_0, \psi^{\rm i}_0, \psi^{\rm r}_1, \psi^{\rm i}_1, 0, 0, 0, 0, \quad 0, 0, 0, 0, 0, 0, 0, 0\right) \tag{20}$$

$$\Psi^{\rm Gr,EM}_{\rm Q2} := \left(0, 0, 0, 0, -\psi^{\rm r}_2, \psi^{\rm i}_2, \psi^{\rm r}_3, \psi^{\rm i}_3, \quad 0, 0, 0, 0, 0, 0, 0, 0\right) \tag{21}$$

$$\Psi^{\rm Gr,EM} := \Psi^{\rm Gr,EM}_{\rm Q1} + \exp\left(i_0 \alpha\right) \Psi^{\rm Gr,EM}_{\rm Q2} \tag{22}$$

the unified description of quantum gravitation and electromagnetism is now proposed as:

$$\left(\nabla^{\rm Gr,EM} - m\right) \Psi^{\rm Gr,EM} = 0 \tag{23}$$

Universal applicability of physical law for different observers ("relativity") is governed by the invariant hypernumber modulus theorem. Applied to

$$\left|\nabla^{\rm Gr,EM} \Psi^{\rm Gr,EM}\right| = \left|m \Psi^{\rm Gr,EM}\right| = |m| \left|\Psi^{\rm Gr,EM}\right| = \left|\nabla^{\rm Gr,EM}\right| \left|\Psi^{\rm Gr,EM}\right| \tag{24}$$

it demands identical values for $|m|$, $\left|\Psi^{\rm Gr,EM}\right|$, and $\left|\nabla^{\rm Gr,EM}\right|$ in equivalent frames of reference, under the influence of any force.

4 Mixing Gravity and Electromagnetism

The transformation $\Psi^{\rm Gr,EM} \rightarrow \Psi^{\rm Gr,EM}\, e^{i_0 \chi}$ leaves the modulus $\left|\Psi^{\rm Gr,EM}\right|$ unchanged and introduces a physical force field A_μ which acts on the particle's electric charge e. Depending on the mixing angle α, the effect of this force is either traditional electromagnetism ($\alpha = \pi/2$), proposed quantum gravity ($\alpha = 0$), or a combination thereof. Since a change in α generally changes $\left|\Psi^{\rm Gr,EM}\right|$, α must remain constant under space-time coordinate transformation. Therefore, there are three constant properties of a spin 1/2 particle in this formulation which characterize its behavior under the influence of a force: m, e, and α.

With a general line element (as in [3], relations 4 through 7)

$$d\tau_{\mathrm{Q1}} := (0, dx_0, 0, 0, 0, 0, 0, 0, \quad 0, 0, 0, 0, 0, 0, 0, 0) \tag{25}$$
$$d\tau_{\mathrm{Q2}} := (0, 0, 0, 0, 0, dx_3, -dx_2, dx_1, \quad 0, 0, 0, 0, 0, 0, 0, 0) \tag{26}$$
$$d\tau_{\mathrm{con16}} := d\tau_{\mathrm{Q1}} + \exp(\alpha i_0)\, d\tau_{\mathrm{Q2}} \tag{27}$$
$$dT := |d\tau_{\mathrm{con16}}| = |d\tau_{\mathrm{Q1}} + \exp(\alpha i_0)\, d\tau_{\mathrm{Q2}}| \tag{28}$$

one can now examine the space-time transformation behavior of a free particle of general α. The conic sedenion modulus $|z|$ of a general sedenion $z = a + \sum_{n=0}^{7} b_n i_n + \sum_{n=0}^{7} c_n \varepsilon_n + d i_0$ is [7]

$$|z| = \sqrt[4]{\left(a^2 + \sum_{n=0}^{7} b_n^2 - \sum_{n=0}^{7} c_n^2 - d^2\right)^2 + 4\left(ad - \sum_{n=0}^{7} b_n c_n\right)^2} \tag{29}$$

and using $|d\vec{x}|^2 := dx_1^2 + dx_2^2 + dx_3^2$ yields:

$$\begin{aligned} |d\tau_{\mathrm{con16}}| &= \sqrt[4]{\left(dx_0^2 + \cos^2\alpha\, |d\vec{x}|^2 - \sin^2\alpha\, |d\vec{x}|^2\right)^2 + 4\left(-\cos\alpha\,\sin\alpha\, |d\vec{x}|^2\right)^2} \\ &= \sqrt[4]{dx_0^4 + |d\vec{x}|^4 + 2dx_0^2\, |d\vec{x}|^2 \left(\cos^2\alpha - \sin^2\alpha\right)} \end{aligned} \tag{30}$$

This relation cannot be represented anymore in a second order tensor formalism (e.g. through the metric tensor $g_{\mu\nu}$ from General Relativity with $d\tau^2 = \sum g_{\mu\nu} dx_\mu dx_\nu$), but contains mixing effects between quantum gravity and electromagnetism that cannot be separated into their constituent forces. It is also distinct from a potential fourth order tensor formalism $d\tau^4 = \sum g^*_{\mu\nu\rho\sigma} dx_\mu dx_\nu dx_\rho dx_\sigma$ due to the non-associativity of conic sedenions.

For all known particles from the *Standard Model* (SM) in physics, the relative strength of the gravitational force is docents of orders of magnitude weaker compared to the electrical force, and writing $\alpha := \frac{\pi}{2} - \beta$ allows to approximate β^2 in the order of the relative force strength $\beta^2 \sim F_{\mathrm{Gr}}/F_{\mathrm{EM}} \sim m^2/e^2$ (F_{Gr} and F_{EM} are the static gravitational and electromagnetic forces between two particles with mass m and electrical charge e; e.g. for two electrons $\beta_{e^-}^2 \sim 10^{-42}$). Using $\gamma := \left(1 - |d\vec{x}|^2 / dx_0^2\right)^{-1/2}$ yields:

$$\begin{aligned} |d\tau_{\mathrm{con16}}|^2_{(\mathrm{SM})} &\approx \sqrt{dx_0^4 + |d\vec{x}|^4 + 2dx_0^2\, |d\vec{x}|^2 \left(\beta^2 - 1\right)} & (31)\\ &\approx \left(dx_0^2 - |d\vec{x}|^2\right) \left(1 + \beta^2 \frac{dx_0^2\, |d\vec{x}|^2}{\left(dx_0^2 - |d\vec{x}|^2\right)^2}\right) & (32)\\ &= dx_0^2 - |d\vec{x}|^2 + (\beta\gamma)^2\, |d\vec{x}|^2 & (33) \end{aligned}$$

This relation could be expressed in second order tensor form $d\tau^2 = \sum g_{\mu\nu} dx_\mu dx_\nu$ as currently used in physics. The factor $(\beta\gamma)^2$ depends on the speed of an observer, which is the result of the two forces gravity and electromagnetism being proposed on different space-time geometries (see in detail [3]). Energy levels for which (33) would exhibit a significant deviation from traditional physics are currently only presumed shortly after the Big Bang. For comparison, the Tevatron accelerator (Fermilab) reaches $\gamma = E_{\mathrm{kin}}/E_0 \approx 10^3$ with protons ($\beta_p^2 \sim 10^{-35}$), and LEP (CERN) reached $\gamma \approx 2 \cdot 10^5$ for electrons ($\beta_{e^-}^2 \sim 10^{-42}$). In both cases there is $(\beta\gamma)^2 \lesssim 10^{-29}$ which makes the traditional length element of a free particle $d\tau^2_{\mathrm{SRT}} = dx_0^2 - |d\vec{x}|^2$ an approximation that is indistinguishable from conic sedenion formulation in these experiments.

5 Conclusion and Outlook

Conic sedenion arithmetic has been used to describe both the classical hyperbolic Dirac equation with electromagnetic field and a counterpart on circular geometry proposed for quantum gravity. The same field A_μ from electromagnetism is also generator of gravitational interaction in this description. The observed difference between the two forces is quantified through a mixing angle α, which becomes a third particle property next to its electrical charge e and mass at rest m. At experimentally accessible energy levels, this formalism approximates to customary second-order tensor form and has α in the order of $\frac{\pi}{2} - \alpha \sim m/e$.

While effects from conic sedenion formulation are eluding current experimental validation, one is able to describe these two forces on the quantum level in a narrow, closed arithmetic, without unsparing use of extra dimensions, fields, or free parameters. Hypothetical particle types which exhibit predominantly gravitational coupling ($\alpha < \pi/4$) could enter into strongly bound states that cannot be broken in current experiments, making its constituents not directly observable. Conic sedenion arithmetic allows to vary the parameter α and therefore is offered for exploration of such states and observables that may indirectly materialize therefrom.

Split-octonions, which are computationally equivalent to hyperbolic octonions, have been shown to describe classical electromagnetism on a non-traditional number system [8]. Description of gravity on an Euclidean background has been called "... the only sane way to do quantum gravity nonperturbatively" [9], and a genuine Euclidean metric (without "Wick rotated" time element $dt \rightarrow i\,dt$) was included in the approach. For both these recent developments, conic sedenions with their circular (Euclidean) and hyperbolic (Minkowski) subalgebras may pose a valid computational tool for further examination, in addition to "asymptotically anti-de Sitter space" suggested by Hawking.

Extension of the invariant hypernumber modulus theorem to other hypernumber types offers mathematical versatility beyond gravitation and electromagnetism. For example, Musès mentioned about w arithmetic that "at least 3 orders of w ($w_{1,2,3}$) exist [...]" which form an anti-commutative 3 cycle [10], similar to a typical representation of SU(2) symmetry currently used for the weak interaction in physics. Another example is the power orbit ε^α which does not fall together with the exponential orbit $\exp \alpha\varepsilon$ of the same base number ε. Current description in physics uses exponential orbits for all forces of the *Standard Model*, whereas power orbits would be the natural transformation with respect to the invariant hypernumber modulus theorem: $\Psi \rightarrow \Psi\,\varepsilon^b$ (b real) leaves $|\Psi|$ unchanged, and so do compound constructs like $\Psi \rightarrow \Psi\,i_0^a \varepsilon^b w^c$ or $\Psi \rightarrow \Psi\,i^a \varepsilon^b w^c$ (a, b, c real). Further investigation into these and similar relations is needed in order to conclusively evaluate hypernumbers for the description of physical law.

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