Hypernumbers and relativity

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Abstract

The Dirac equation in physics was described earlier in a simple form through hyperbolic octonion arithmetic, and a circular octonion counterpart exhibited certain behavior which one might expect from a quantum gravitational primitive. Both relations are expressible with use of conic sedenion arithmetic (M-algebra) as the unifying number concept. A general concern exists when proposing physical forces on the same space-time, but with respect to different underlying geometries. Therefore, a needed concept of relativity will now be suggested in terms of an invariant hypernumber modulus, to warrant universal applicability of physical law in equivalent frames of reference. To support validity of such a concept, conic sedenions will be analyzed with respect to their hyperbolic and circular subalgebras. Representing classical Minkowski and a new Euclidean space-time metric respectively, an alignment program for large body (non-quantum) physics will be proposed. Translated into the language and concepts from General Relativity, its effective equivalence in the description of gravity will be shown. Some approaches to physics on number systems other than traditional complex numbers will briefly be compared to the current investigation in this paper.

Key words: hypernumbers, special relativity, general relativity, conic complex numbers, countercomplex numbers, sedenions, octonions, physics on hyperbolic and circular geometry, quantum gravity

1 Introduction

Any law in physics must be governed by a principle of relativity: After specifying equivalent frames of reference ("lab frames"), an invariance condition must warrant universal validity of such law. When measurements first indicated over 100 years ago that the speed of light does not vary with the relative speed between light source (the Sun) and observer (an interferometer on our rotating Earth), Albert Einstein proposed the theory of *Special Relativity* [1]. It identifies conditions under which lab frames are equivalent, and how they relate through space-time transformations. Initially based on a thought experiment only [2], Einstein then extended this equivalence to the accelerated observer of *General Relativity*: Since gravitation affects all forms of energy alike, including a body's mass at rest, any two gravitationally free-falling observers must pose equivalent frames of reference.

Special Relativity is probably the most precisely confirmed theory in physics, on both large scales and the quantum level. *General Relativity* also enjoys unrefuted validity on large scales, from the size of a rock (or objects of comparable energy) up to the entire universe. On the quantum level, however, inherent computational difficulty from its mathematical description poses some unresolved problems to-date.

While no quantum gravitational effects have been conclusively measured yet, current investigation on conic sedenion arithmetic [3,4] into signs of gravity [5,6] must test its compatibility with *General Relativity*, despite the fundamental difference in mathematical and conceptual description.

Therefore, relativity will first be defined by demanding invariance of a hypernumber modulus. Applied to conic sedenion algebra, the circular and hyperbolic octonion subalgebras can be related to both the classical Minkowski space-time metric from *Special Relativity* and a new Euclidean space-time metric. A "natural alignment of elementary equations" (*NatAliE* equations) program will be defined and proposed to integrate gravity on Euclidean metric into classical physics on Minkowski space-time. Translated into language and concepts from *General Relativity*, this *NatAliE* equations program will effectively turn out to be equivalent in its description of gravity.

At the end, a brief comparison with some other approaches to physics on non-traditional number systems will be offered.

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2 Invariant Conic Sedenion Modulus as Concept for Relativity

As shown in [5,6], using mass m and partial derivatives $\partial_\mu := \partial/\partial x_\mu$ of space $\vec{x} := (x_1, x_2, x_3)$ and time [1] $t := x_0$, the following conic sedenions [2] to the basis $b_{\text{con16}} \in \{1, i_1, ..., i_7, i_0, \varepsilon_1, ..., \varepsilon_7\}$

$$\triangledown_{\text{Q1}} := (-m, \partial_0, 0, 0, 0, 0, 0, 0, \quad 0, 0, 0, 0, 0, 0, 0, 0) \tag{1}$$
$$\triangledown_{\text{Q2}} := (0, 0, 0, 0, 0, \partial_3, -\partial_2, \partial_1, \quad 0, 0, 0, 0, 0, 0, 0, 0) \tag{2}$$

and a real-number mixing angle α allow to model an operator $\triangledown_{\text{con16}}$ to act on a particle's wave function:

$$\triangledown_{\text{con16}} := \triangledown_{\text{Q1}} + \exp(\alpha i_0)\, \triangledown_{\text{Q2}} \tag{3}$$

The exponent term $\exp(\alpha i_0)$ effectively rotates $\triangledown_{\text{Q2}}$ in the $(1, i_0)$ plane and allows to transition the classical Dirac equation in physics from a hyperbolic octonion formulation into a new circular octonionic counterpart.

Physical frames of reference will now be defined as equivalent if the coordinate transformation from one into another leaves the modulus dT of a new space-time sedenion $d\tau_{\text{con16}}$ invariant.

From $\triangledown_{\text{con16}}$ (3) the positions of spacial dimensions and time with respect to the current conic sedenion algebra can be identified. The following definitions use small variations dx_μ:

$$d\tau_{\text{Q1}} := (0, dx_0, 0, 0, 0, 0, 0, 0, \quad 0, 0, 0, 0, 0, 0, 0, 0) \tag{4}$$
$$d\tau_{\text{Q2}} := (0, 0, 0, 0, 0, dx_3, -dx_2, dx_1, \quad 0, 0, 0, 0, 0, 0, 0, 0) \tag{5}$$
$$d\tau_{\text{con16}} := d\tau_{\text{Q1}} + \exp(\alpha i_0)\, d\tau_{\text{Q2}} \tag{6}$$

The corresponding modulus is:

$$dT := |d\tau_{\text{con16}}| = |d\tau_{\text{Q1}} + \exp(\alpha i_0)\, d\tau_{\text{Q2}}| \tag{7}$$

One can then formulate a hypernumber modulus invariance theorem as follows:

Theorem 1 *Two lab frames A and A' are equivalent with respect to physical law if a linear transformation from the respective x_μ into x'_μ coordinates leaves dT from (7) invariant. This definition of relativity is consistent with current description in physics.*

To provide proof, it will first be shown that without gravity the invariance condition on dT describes all central principles of *Special Relativity* (Minkowski space-time, and mass-energy-momentum relation). Then, an alignment program will be developed for large body (non-quantum) physics that allows to translate the hypernumber relations into the language and concepts of *General Relativity*, and to prove its effective equivalence.

3 Equivalence to Special Relativity when Excluding Gravity

The following physical principles govern *Special Relativity* (see e.g. [7] chapters 1 through 5, or [8] chapter 13):

(1) Any two frames of reference that are in constant (non-accelerated) motion with respect to each other, or in no relative motion at all, at any place in space $\vec{x} := (x_1, x_2, x_3)$ and time $t := x_0$, are equivalent.
(2) The speed of light c is constant in any equivalent frame of reference. The equation of motion for light (in the current choice of $c \equiv 1$) is described as $dt^2 - |d\vec{x}|^2 = 0$. In general, for any two equivalent frames of reference A and A' the transformation from respective x_μ into x'_μ coordinates leaves the expression $d\tau^2_{\text{SRT}} := dt^2 - |d\vec{x}|^2 = dt'^2 - |d\vec{x}'|^2$ invariant (for light: $d\tau_{\text{SRT}} = 0$). This is also called Minkowski space-time (metric).
(3) A body's mass at rest m is a form of energy E. They relate to momentum $\vec{p} := (p_1, p_2, p_3)$ through $m^2 = E^2 - |\vec{p}|^2$ which also remains invariant for equivalent frames of reference.

[1] Physical constants c, h, and G are set to 1 in this paper, since they are non-essential for the mathematical structure. This procedure is also common in related fields of physics. In addition, all indices will be written as lower indices here; if present, a metric tensor will be written explicitely. This deviation from common notation will later avoid a potential ambiguity in expressions on Euclidean (circular) and Minkowski (hyperbolic) space-time metric. Hypernumber notation and definitions are carried forward from [5,6].

[2] Please note that definition (4) in [5] is incorrect and should be $\nabla_{\text{hyp8}} := (-m, \partial_0, 0, 0, 0, -\partial_3, \partial_2, -\partial_1)$, as well as definition (3) should be $\Psi_{\text{hyp8}} := \left(\psi^{\text{r}}_0, \psi^{\text{i}}_0, \psi^{\text{r}}_1, \psi^{\text{i}}_1, \psi^{\text{r}}_2, -\psi^{\text{i}}_2, -\psi^{\text{r}}_3, -\psi^{\text{i}}_3\right)$. The different definitions were the result of using a conic sedenion multiplication table which identified the classical octonion element "l" with sedenion element $-i_4$ instead of i_4, therefore not being consistent with the cited sources [3,13].

Principles (1) and (2) are geometrical conditions on space and time, whereas (3) is a relation in terms of physical properties energy and momentum.

Lemma 2 *With only its hyperbolic octonion subalgebra, relativity as defined through the invariant conic sedenion modulus dT reduces to the space-time relations from the theory of Special Relativity (principles 1 and 2).*

PROOF. The hyperbolic octonion subalgebra from (7) to the basis $b_{\text{hyp8}} \in \{1, i_1, i_2, i_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7\}$ corresponds to $\alpha_{\text{hyp8}} = \pi/2$. With $\exp(\alpha_{\text{hyp8}} i_0) = i_0$ and sedenion modulus from [3,4] one obtains:

$$
\begin{aligned}
dT_{\text{hyp8}} &= |d\tau_{\text{Q1}} + i_0 d\tau_{\text{Q2}}| \\
&= |(0, dx_0, 0, 0, 0, 0, 0, 0, \quad 0, 0, 0, 0, 0, -dx_3, dx_2, -dx_1)| \\
&= \sqrt[4]{(dx_0^2 - dx_3^2 - dx_2^2 - dx_1^2)^2} \\
&= \sqrt{dt^2 - |d\vec{x}|^2}
\end{aligned} \tag{8}
$$

Demanding invariance of dT_{hyp8} under a coordinate transformation of the x_μ therefore corresponds to the invariance condition $d\tau_{\text{SRT}}^2 := dt^2 - |d\vec{x}|^2$ from *Special Relativity* and reproduces principle (2) above.

In order to satisfy principle (1), such a coordinate transformation between x_μ and x'_μ coordinates of two equivalent frames of reference A and A' must exist, and be a function of constant relative speed between A and A' only. The so-called Lorentz transformation in physics satisfies this requirement, in addition to leaving $d\tau_{\text{SRT}}^2 := dt^2 - |d\vec{x}|^2$ invariant (see e.g. the first chapters in [7] for a comprehensive introduction, or [8] section 13.4). This proves lemma 2.

The relation $m^2 = E^2 - |\vec{p}|^2$ from principle (3) can be shown to be a necessary condition when consistently defining dynamical interaction through physical forces (see e.g. [7] chapter 6). This requires introduction of additional concepts from physics.

But also if treating $E \equiv p_0$ and $\vec{p}$ as mere Fourier coefficients of time $t \equiv x_0$ and space $\vec{x}$ respectively (see also [6], or [9] equation 75.8)

$$
\psi(x) = \int \frac{d^4p}{(2\pi)^4} \psi(p) \exp\left[i \sum_{\nu=0}^{3} p_\nu x_\nu\right] \tag{9}
$$

without further physical interpretation, one can show that the modulus of a hyperbolic octonion containing these p_ν reproduces this important fundamental relation in physics, too [3].

Lemma 3 *Formulation of relativity through hyperbolic octonions produces a result from classical physics, where the four-vectors $x = (t, \vec{x})$ and $p = (E, \vec{p})$ relate to each other through Fourier transformation, and satisfy corresponding invariance conditions $d\tau_{\text{SRT}}^2 = dt^2 - \left|d\vec{x}\right|^2$ and $m^2 = E^2 - |\vec{p}|^2$ (principle 3).*

PROOF. In order to conclude from an invariance condition on space-time dT_{hyp8} to an invariant energy-momentum modulus M_{hyp8}, the derivatives of space and time ∂_μ from $\bigtriangledown_{\text{con16}}$ (3) will be related to Fourier coefficients momentum $\vec{p} := (p_1, p_2, p_3)$ and energy $E := p_0$ respectively. As opposed to the classical case, special consideration must be taken in hyperbolic octonion representation with respect to the chosen exponential orbit.

In the definition of Ψ_{hyp8} [5] the traditional imaginary base i was identified with the sedenion base i_1 when assigning the real and imaginary parts ψ_μ^{r} and ψ_μ^{i} of the classical wave function elements ψ_μ to coefficients on an octonion base.

This identification will be carried forward, and each ψ_μ is translated with (9) separately, in direct analogy to classical physics:

$$
\psi_\mu(x) = \int \frac{d^4p}{(2\pi)^4} \psi_\mu(p) \exp\left[i_1 \sum_{\nu=0}^{3} p_\nu x_\nu\right] \tag{10}
$$

Since the ψ_μ only contain a real part ψ_μ^{r} and imaginary part ψ_μ^{i} to the basis i_1, the exponential term $\exp\left[i_1 \sum_{\nu=0}^{3} p_\nu x_\nu\right]$ commutes with the $\psi_\mu(p)$, and multiplication with additional sedenion base elements will be associative (conic sedenions are alternative [3]).

[3] An implicit assumption is made that such expression through Fourier coefficients is actually possible, so that $\partial_0 \psi(x) = iE\, \psi(x)$ and $\partial_j \psi(x) = ip_j\, \psi(x)$ allows to retrieve these coefficients. This is in analogy to current description in quantum mechanics, where $i\partial_0$ and $-i\partial_j$ are called “operators” on the wave function ψ to retrieve “measurables” E and $\vec{p}$ (but note the sign convention of the ∂_0 and E terms).

Setting $\alpha_{\mathrm{hyp8}} = \pi/2$ in (3) yields

$$\begin{aligned}\triangledown_{\mathrm{hyp8}} &= (m, \partial_0, 0,0,0,0,0,0, \quad 0,0,0,0,0,-\partial_3, \partial_2, -\partial_1) \\ &= m + i_1\partial_0 - \varepsilon_5\partial_3 + \varepsilon_6\partial_2 - \varepsilon_7\partial_1\end{aligned} \tag{11}$$

and one can identify for the four components ∂_μ in $\triangledown_{\mathrm{hyp8}}$ to their respective bases:

$$(i_1\partial_0)\,\psi_\mu(x) = i_1\,(i_1 E)\,\psi_\mu(x) = -E\,\psi_\mu(x) \tag{12}$$
$$(-\varepsilon_5\partial_3)\,\psi_\mu(x) = -\varepsilon_5\,(i_1 p_3)\,\psi_\mu(x) = \varepsilon_4 p_3\,\psi_\mu(x) \tag{13}$$
$$(\varepsilon_6\partial_2)\,\psi_\mu(x) = \varepsilon_6\,(i_1 p_2)\,\psi_\mu(x) = \varepsilon_7 p_2\,\psi_\mu(x) \tag{14}$$
$$(-\varepsilon_7\partial_1)\,\psi_\mu(x) = -\varepsilon_7\,(i_1 p_1)\,\psi_\mu(x) = \varepsilon_6 p_1\,\psi_\mu(x) \tag{15}$$

The "operator" $\triangledown_{\mathrm{hyp8}}$ (without m, which is constant) therefore relates to the following "observable" p_{hyp8}:

$$\begin{aligned}p_{\mathrm{hyp8}} &= -E + \varepsilon_4 p_3 + \varepsilon_6 p_1 + \varepsilon_7 p_2 \\ &= (-E, 0,0,0,0,0,0,0, \quad 0,0,0,0,p_3,0,p_1,p_2)\end{aligned} \tag{16}$$

Replacing operator $\triangledown_{\mathrm{hyp8}}$ with observable p_{hyp8}, the hyperbolic Dirac equation $\triangledown_{\mathrm{hyp8}}\Psi_{\mathrm{hyp8}} = 0$ [5] can then be written as

$$p_{\mathrm{hyp8}}\Psi_{\mathrm{hyp8}} = m\Psi_{\mathrm{hyp8}} \tag{17}$$

(m real), and modularity of the number system allows to identify for non-zero $|\Psi_{\mathrm{hyp8}}|$:

$$\begin{aligned}|p_{\mathrm{hyp8}}\Psi_{\mathrm{hyp8}}| = |p_{\mathrm{hyp8}}|\,|\Psi_{\mathrm{hyp8}}| &= |m\Psi_{\mathrm{hyp8}}| = |m|\,|\Psi_{\mathrm{hyp8}}| \\ |p_{\mathrm{hyp8}}| &= |m|\end{aligned} \tag{18}$$

The hyperbolic octonion modulus M_{hyp8} of p_{hyp8} is

$$M_{\mathrm{hyp8}} := |p_{\mathrm{hyp8}}| = \sqrt{E^2 - |\vec{p}|^2} = |m| \tag{19}$$

which reproduces the classical mass-energy-momentum relation $m^2 = E^2 - |\vec{p}|^2$. This proves lemma 3.

Therefore, it has been shown that key fundamental principles from *Special Relativity* are mathematically described through the hyperbolic octonion subalgebra of the invariance condition on $dT := |d\tau_{\mathrm{con16}}|$. This includes Minkowski space-time and the mass-energy-momentum relation $m^2 = E^2 - |\vec{p}|^2$. It is noted that m, E and $\vec{p}$ were used only as constants, and no physical interpretation as mass, energy or momentum was needed.

4 Equivalence to General Relativity when Including Gravity

The invariant conic sedenion modulus $|d\tau_{\mathrm{con16}}|$ in its general form (i.e. for any α) has been suggested to also describe the gravitational force, for which current and experimentally proven description in physics is given by the theory of *General Relativity*. When probing this suggestion for validity, special consideration must be taken with respect to the following:

(1) The theory of *General Relativity* is built on principles that don't exhibit an immediate relation to hypernumber arithmetic.
(2) Relativity as defined through an invariant hypernumber modulus has been developed from a quantum physical formalism, whereas computational obstacles in *General Relativity* limit conclusive quantum gravitational calculations.

To avoid potential ambiguity or speculations with respect to quantum gravity, the following proof of equivalence will be limited to the branch in physics that has been experimentally verified: Large body (non-quantum) gravitation.

The following will be demonstrated:

(1) For large bodies in physics, the gravitational force can be treated separately from other forces. Therefore, existence of a "Natural Alignment of Elementary Equations" (*NatAliE* equations) program will be proposed that allows to project purely gravitational effects from the invariant conic sedenion modulus ($\alpha = 0$) onto a non-gravitational observer ($\alpha = \pi/2$).
(2) These effects will be applied to a known equation of motion of a non-gravitational observer. The so modified equation of motion will prove equivalent to the linearized field equations from *General Relativity*.

(3) A known "bootstrap" argumentation will be referred to, which has been shown to lead from the linearized field equations to the covariant field equations in *General Relativity* through consistent self-interaction of the gravitational field. This reference will conclude the proof.

In the absence of a current, generally agreed-upon quantum theory of gravitation, this approach appears to be reasonable and sufficient for non-quantum physics. Since it cannot be verified due to lack of empirical validation and mathematical description thereof, the following proposition must be added as prerequisite for the proof:

Proposition 4 *For large body (non-quantum) physics a "Natural Alignment of Elementary Equations" (NatAliE Equations) program is possible that calculates purely gravitational effects separately from other physical forces and effects. The gravitational effects will be deduced from the circular octonion subalgebra in* $|d\tau_{\text{con16}}|$*, all other effects from the hyperbolic octonion subalgebra. When an equation of motion is expressed in a way that unambiguously separates terms which originate from the gravitational force from any other terms, one can project results from circular octonion (Euclidean) space-time onto otherwise hyperbolic octonion (Minkowski) formulations. This correctly describes gravity within said scope.*

The following statements support the argument for this proposition:

(1) The known ability of a quantum system to be in more than one distinct state simultaneously - prior to measurement - is reflected in the continuous "mixing angle" α in $d\tau_{\text{con16}}$. Transition to large body (non-quantum) physics removes this degree of freedom, and particles and forces must either act with respect to the distinct $\alpha = 0$ or $\alpha = \pi/2$ space-time.
(2) The circular octonion subalgebra of $\bigtriangledown_{\text{con16}}\Psi_{\text{con16}} = 0$ has exhibited a signature of gravity (treating particles and anti-particles alike [6]), which is a behavior distinct from the hyperbolic octonion subalgebra that was proven above to be equivalent to *Special Relativity* (excluding gravity).
(3) It is possible that current description of gravity in *General Relativity* evolved from a traditional point of view that describes gravitational measurements and descriptions as variations of the space-time that governs the electromagnetic force: Hyperbolic Minkowski space-time predominates human everyday perception of a world made from atoms (electrons and nuclei in electromagnetic interaction) and light (electromagnetic radiation).
(4) Therefore, expressing space-time in both circular (Euclidean) and hyperbolic metric in parallel, and then projecting the relations from circular geometry onto hyperbolic geometry, appears to be a valid mathematical procedure to investigate gravity in conic sedenions.

It is concluded that proposition 4 is justifiable with respect to the current state of mathematical description of physical law, to be empirically verified or falsified in the future.

4.1 The "Natural Alignment of Elementary Equations" (NatAliE Equations) Program

Relations from the circular octonion subalgebra ($\alpha = 0$, $\exp i_0\alpha = 1$) contained in the invariance condition on the conic sedenion modulus $|d\tau_{\text{con16}}|$ can be calculated by comparison with the corresponding hyperbolic octonion subalgebra above ($\alpha = \pi/2$, $\exp i_0\alpha = i_0$).

In analogy to (8) one obtains

$$\begin{aligned} dT_{\text{cir8}} &= |d\tau_{\text{Q1}} + d\tau_{\text{Q2}}| \\ &= |(0, dx_0, 0, 0, 0, dx_3, -dx_2, dx_1, \quad 0, 0, 0, 0, 0, 0, 0, 0)| \\ &= \sqrt{dt^2 + |d\vec{x}|^2} \end{aligned} \tag{20}$$

and calculations similar to the ones leading to (19) yield:

$$M_{\text{cir8}} := |p_{\text{cir8}}| = \sqrt{E^2 + |\vec{p}|^2} \tag{21}$$

This new property [4] will now be called a body's "circular mass", as opposed to a body's "hyperbolic mass" M_{hyp8}.

In *Special Relativity* the classical ("hyperbolic") Lorentz transformation warrants invariance of terms dT_{hyp8} and M_{hyp8} for frames of reference that are in non-accelerated relative motion (e.g. [8] section 13.4), or no relative motion at all. Using relative

[4] By comparison with the classical (hyperbolic) Dirac equation, M_{hyp8} was earlier identified with the mass at rest m of a body. One may now argue that M_{cir8} corresponds to a body's heavy mass (which gravitationally generates weight) since it is defined on purely gravitational circular (Euclidean) space-time; whereas M_{hyp8} may correspond to a body's inert mass (i.e. its resistance to acceleration). While this physical interpretation may be possible, the descriptive symbols M_{hyp8} and M_{cir8} will still be used to highlight the notion that both are limited projections of a wider concept $M_{\text{con16}} := |p_{\text{con16}}|$.

speed $\vec{v}$ between two such lab frames A and A' with respective coordinates x_μ and x'_μ, a new "circular" Lorentz transformation Λ_{cir8} will now be defined:

$$\Lambda_{\text{cir8}} := \begin{pmatrix} \left(1+|\vec{v}|^2\right)^{-1/2} & |\vec{v}|\left(1+|\vec{v}|^2\right)^{-1/2} & 0 & 0 \\ -|\vec{v}|\left(1+|\vec{v}|^2\right)^{-1/2} & \left(1+|\vec{v}|^2\right)^{-1/2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{22}$$

$$x'_\mu = \sum_{\nu=0}^{3} \left(\Lambda_{\text{cir8}}\right)_{\mu\nu} x_\nu$$

The x_1 axis in this representation is oriented per definition in the direction of the connecting vector between each two attracting masses A and A' (or a mass in A that generates a gravitational force on an observer in A'), which is generally different from the direction of relative motion $\vec{v}$. This definition of orientation narrows the frame of reference concept here to pairwise gravitational interaction: The direction affected by the circular Lorentz transformation Λ_{cir8} depends on where the observer in A' is located relative to the gravity generating mass in A. It can therefore be called a "local" transformation, whereas the classical Lorentz transformation is defined globally (for any mass distribution at any x_μ) by orienting the x_1 axis in the direction of relative motion $\vec{v}$ of the observer. Nevertheless, moduli are magnitude without orientation and remain invariant globally in both cases.

Lemma 5 *For two frames of reference that are in constant (non-accelerated) motion with respect to each other, or no relative motion at all, at any place in space $\vec{x} := (x_1, x_2, x_3)$ and time $t := x_0$, the circular Lorentz transformation leaves the circular octonion modulus $dT_{\text{cir8}} = \sqrt{dt^2 + |d\vec{x}|^2}$ invariant.*

PROOF. The transformation (22) on $|x| = \sqrt{t^2 + |\vec{x}|^2} = \sqrt{\sum_{\mu=0}^{3} x_\mu^2}$ yields:

$$\begin{aligned} |x'| &= \sqrt{\sum_{\mu=0}^{3}\left(\sum_{\nu=0}^{3}\left(\Lambda_{\text{cir8}}\right)_{\mu\nu} x_\nu\right)^2} \\ &= \sqrt{\left(1+|\vec{v}|^2\right)^{-1}\left(x_0^2 + 2|\vec{v}|\,x_0x_1 + |\vec{v}|^2 x_1^2\right) + \left(1+|\vec{v}|^2\right)^{-1}\left(|\vec{v}|^2 x_0^2 - 2|\vec{v}|\,x_0x_1 + x_1^2\right) + x_2^2 + x_3^2} \\ &= \sqrt{\sum_{\mu=0}^{3} x_\mu^2} = \sqrt{t^2 + |\vec{x}|^2} \end{aligned} \tag{23}$$

Since Λ_{cir8} is only a function of relative speed $\vec{v}$, it is valid at any position in space and time. Because $\vec{v}$ is constant (no relative acceleration; $d\vec{v} = 0 \Longrightarrow d\Lambda_{\text{cir8}} = 0$) it also leaves

$$\begin{aligned} dT_{\text{cir8}} = |dx'| &= \sqrt{\sum_{\mu=0}^{3} dx'^2_\mu} \\ &= \sqrt{\sum_{\mu=0}^{3}\left(\sum_{\nu=0}^{3}\left(\Lambda_{\text{cir8}}\right)_{\mu\nu} dx_\nu\right)^2} \end{aligned} \tag{24}$$

invariant. This proves lemma 5.

The identity

$$\frac{M_{\text{cir8}}}{M_{\text{hyp8}}} = \frac{dT_{\text{cir8}}}{dT_{\text{hyp8}}} \tag{25}$$

derived from the Fourier coefficient pairs $(E, |\vec{p}|)$ and $(t, |\vec{x}|)$, together with relative speed $|\vec{v}| = |d\vec{x}|/dt$ between A and A', yields a projected value of M_{cir8} in hyperbolic space-time:

$$\frac{M_{\text{cir8}}}{M_{\text{hyp8}}} = \sqrt{\frac{E^2 + |\vec{p}|^2}{E^2 - |\vec{p}|^2}} = \sqrt{\frac{1 + |\vec{v}|^2}{1 - |\vec{v}|^2}} \tag{26}$$

$$M_{\text{cir8}} = M_{\text{hyp8}} \sqrt{\frac{1 + |\vec{v}|^2}{1 - |\vec{v}|^2}} \tag{27}$$

The mass term M_{cir8}, which is to be used in purely gravitational context only, is increased as compared to M_{hyp8}.

A consequence of the circular Lorentz transformation is that distances $|\vec{x}'|$ as observed in A' appear expanded to

$$|\vec{x}| = |\vec{x}'| \sqrt{1 + |\vec{v}|^2} \tag{28}$$

for an observer in A, if this distance is parallel to the connecting vector between A and A'. Distances perpendicular to the connecting vector between A and A' remain unchanged. This length expansion is the counterpart of classical length contraction from the hyperbolic Lorentz transformation, for which distances subject to length contraction are parallel to the vector of relative motion $\vec{v}$. The difference between length contraction and length expansion, as well as, the difference in spacial orientation of affected distances versus invariant measures in the perpendicular planes, will need to be taken into account when projecting gravity from circular (Euclidean) space-time onto hyperbolic (Minkowski) spacetime.

In order to execute proposition 4 one needs to bring an equation of motion in large body (non-quantum) physics into a form that unambiguously separates terms originating from the gravitational force from any other terms, and then perform above projections from circular (Euclidean) onto hyperbolic (Minkowski) space-time. This defines the *NatAliE* equations program.

4.2 Prerequisites from Physics

The following are known relations in physics, together with some immediate deductions therefrom, to be used to prove equivalence of the *NatAliE* equations program with *General Relativity* for non-quantum physics. Unless otherwise specified, all following physical terms and properties (like Λ_{hyp8}, $T_{\mu\nu}$, $M_{\mu\nu}$, u_ν, ρ, $g_{\mu\nu}$, $h_{\mu\nu}$, or γ) are functions of time and space x_μ ($\mu = 0, 1, 2, 3$).

4.2.1 Tensors and the Classical (Hyperbolic) Lorentz Transformation

The classical hyperbolic Lorentz transformation Λ_{hyp8} from x into x' coordinates can be expressed in symmetric 4×4 matrix form (e.g. [7], or [8] section 13.4):

$$(\Lambda_{\text{hyp8}})_{\nu\mu} = (\Lambda_{\text{hyp8}})_{\mu\nu} = \frac{\partial x_\mu}{\partial x'_\nu} = \frac{\partial x_\nu}{\partial x'_\mu} \tag{29}$$

A tensor of second order $T_{\mu\nu}$ satisfies the following transformation rule from x into x' coordinates per definition:

$$\begin{aligned} T'_{\mu\nu} &= \sum_{\rho,\sigma=0}^{3} \left(\frac{\partial x_\rho}{\partial x'_\mu}\right)\left(\frac{\partial x_\sigma}{\partial x'_\nu}\right) T_{\rho\sigma} \\ &= \sum_{\rho,\sigma=0}^{3} (\Lambda_{\text{hyp8}})_{\rho\mu} (\Lambda_{\text{hyp8}})_{\sigma\nu} T_{\rho\sigma} \end{aligned} \tag{30}$$

In abbreviated matrix arithmetic this can be written as:

$$\begin{aligned} T' &= \Lambda_{\text{hyp8}} \Lambda_{\text{hyp8}}^{\text{T}} T \\ &= \Lambda_{\text{hyp8}}^2 T \end{aligned} \tag{31}$$

As a special case, if a physical relation can be brought through a certain choice of coordinates into a form that uses the Kronecker symbol $\delta_{\mu\nu}$, one can generalize this physical relation by interpreting $\delta_{\mu\nu}$ as a coordinate-dependent representation of an otherwise general tensor:

$$\begin{aligned} \delta_{\mu\nu} &\mapsto \sum_{\rho,\sigma=0}^{3} (\Lambda_{\text{hyp8}})_{\rho\mu} (\Lambda_{\text{hyp8}})_{\sigma\nu} \delta_{\rho\sigma} \\ &= \left(\Lambda_{\text{hyp8}}^2\right)_{\mu\nu} \end{aligned} \tag{32}$$

The square of the classical Lorentz transformation Λ^2_{hyp8} is the tensor that generalizes $\delta_{\rho\sigma}$ for any equivalent frame of reference with respect to *Special Relativity.*

For tensor calculus in general, if a linear relation between tensors is valid in one particular choice of coordinates x_μ (or frame of reference A), its validity can automatically be inferred for any equivalent coordinate system x'_μ (or frame of reference A').

4.2.2 Speed, Four-Velocity, and Explicit Form of the Classical (Hyperbolic) Lorentz Transformation

The speed $\vec{v}$ of a physical body is its propagation in space $d\vec{x}$ during a particular time interval $dt \equiv dx_0$:

$$\vec{v} = \frac{d\vec{x}}{dx_0} \tag{33}$$

This definition of speed generally transforms non-linear from one frame of reference A into another A' (see e.g. [8] section 13.5). Therefore, a four-velocity (or "invariant speed") u_μ is defined as

$$u_\mu := \frac{dx_\mu}{dx_0}\left(1 - \frac{|d\vec{x}|^2}{dx_0^2}\right)^{-1/2} \tag{34}$$

which transforms from A into A' coordinates through the linear Lorentz transformation:

$$u'_\mu = \sum_{\nu=0}^{3} (\Lambda_{\text{hyp8}})_{\mu\nu}\, u_\nu \tag{35}$$

The classical Lorentz transformation and its square can then be written as [5]:

$$\Lambda_{\text{hyp8}} = \begin{pmatrix} u_0 & u_1 & 0 & 0 \\ u_1 & u_0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{36}$$

$$\Lambda^2_{\text{hyp8}} = \begin{pmatrix} u_0^2 + u_1^2 & 2u_0u_1 & 0 & 0 \\ 2u_0u_1 & u_0^2 + u_1^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{37}$$

4.2.3 The Energy-Momentum Tensor without Pressure ("Mass Tensor")

The field equations of *General Relativity* are expressed with the use of an energy-momentum tensor that contains terms from the motion of an arbitrary mass distribution, together with hydrostatical pressure p. For ideal gases and many highly compressed hot bodies, p can be modeled from statistical random motion of individual particulates. In all other cases, it is an aggregate physical property that is internally realized through quantum mechanical interaction between its constituents (like e.g. a quark plasma during supernova explosion). Since quantum mechanical effects are excluded here, it will be assumed that any pressure term could be realized by statistical random motion of many individual particulates, and therefore it will be set to $p = 0$ further on [6].

[5] While these expressions are in coordinates for which the x_1 axis is oriented in the direction of relative motion between A and A', it is noted that a general form can be obtained for any coordinate system. Every general Lorentz transformation $\bar{\Lambda}_{\text{hyp8}}$ and its square $\bar{\Lambda}^2_{\text{hyp8}}$ can be brought into the above form through a rotation in space α: $\Lambda_{\text{hyp8}} := \alpha\bar{\Lambda}_{\text{hyp8}}\alpha^{-1}$. Since $\bar{\Lambda}^{\mathrm{T}}_{\text{hyp8}} = \bar{\Lambda}_{\text{hyp8}}$ and $\alpha^{-1} = \alpha^{\mathrm{T}}$ one obtains $\Lambda^{\mathrm{T}}_{\text{hyp8}} = \left(\alpha\bar{\Lambda}_{\text{hyp8}}\alpha^{-1}\right)^{\mathrm{T}} = \Lambda_{\text{hyp8}}$ and therefore $\Lambda^2_{\text{hyp8}} = \Lambda_{\text{hyp8}}\Lambda^{\mathrm{T}}_{\text{hyp8}} = \alpha\bar{\Lambda}_{\text{hyp8}}\bar{\Lambda}_{\text{hyp8}}\alpha^{-1} = \alpha\bar{\Lambda}^2_{\text{hyp8}}\alpha^{-1}$. In general, $\bar{\Lambda}_{\text{hyp8}}$ transforms the same as $\bar{\Lambda}^2_{\text{hyp8}}$ under any rotation in space α, and one can continue to examine the special case of above coordinate orientation without losing generality.

[6] This approach is correct if one can show that all remaining terms are equivalent with *General Relativity* for any amount of sources of gravitation; including the 1 large point mass approximation for calculating the motion of a planet around a star, or ficticious calculations for all 10^{60} ($\pm$ a few) atoms in that same star.

Omitting hydrostatical pressure, the energy-momentum tensor (e.g. [8] relation 18.72, or [10] relation 13.7) reduces to the mass tensor $M_{\mu\nu}$. Using mass density ρ and four-velocity u_μ from (34) the mass tensor is:

$$M_{\mu\nu} := \rho u_\mu u_\nu \tag{38}$$

All terms ρ and u_μ are a function of space and time, and the resulting mass tensor $M_{\mu\nu}$ can therefore describe an arbitrary distribution of masses, which may be in any relative motion with respect to each other. For a single point mass (or particulate) $\rho_{(l)}$ the mass tensor $M_{(l)\mu\nu}$ can be brought through rotation in space into the form

$$M_{(l)\mu\nu} = \begin{pmatrix} u_0^2 & u_0 u_1 & 0 & 0 \\ u_0 u_1 & u_1^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{39}$$

because the general four-velocity field u_μ reduces in this case to the distinct four-velocity of the point mass $\rho_{(l)}$.

With use of the Minkowski metric tensor $\eta_{\mu\nu}$ which is defined as

$$\eta_{\mu\nu} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{40}$$

and Λ^2_{hyp8} from (37) the following identities can be verified for a point mass $\rho_{(l)}$:

$$\sum_{\mu,\nu=0}^{3} M_{(l)\mu\nu}\eta_{\mu\nu} = \rho_{(l)} \left(u_0^2 - u_1^2\right) = \rho_{(l)} \tag{41}$$

$$2M_{(l)\mu\nu} - \rho_{(l)}\eta_{\mu\nu} = \rho_{(l)} \left(\Lambda^2_{\text{hyp8}}\right)_{\mu\nu} \tag{42}$$

Since $M_{(l)\mu\nu}$, $\eta_{\mu\nu}$, and $\left(\Lambda^2_{\text{hyp8}}\right)_{\mu\nu}$ are tensors and $\rho_{(l)}$ is a constant, the relation (42) is valid for any mass distribution $M_{\mu\nu}$: The transformation rules for all terms in this linear equation are the same, warranted through tensor calculus, and each point mass can be transformed into another equivalent frame of reference, using classical hyperbolic Lorentz transformation and a rotation in space, in a way that sets the direction of motion of another point mass $\rho_{(l+1)}$ in x_1 direction. Infinite iteration over infinitesimal masses $\rho_{(l)}$ therefore describes any mass density distribution:

$$2M_{\mu\nu} - \rho\eta_{\mu\nu} = \rho \left(\Lambda^2_{\text{hyp8}}\right)_{\mu\nu} \tag{43}$$

4.2.4 Newton Gravity Generalized According to Special Relativity Rules

In [11] chapter 7 ("Incompatibility of Gravity and Special Relativity") offers a detailed analysis on how expression of gravity using a non-accelerated observer leads to an incorrect description of gravity. The hypernumber invariance theorem under investigation in this paper shares some core assumptions with this analysis, but applies modifications from projecting gravitational terms from circular onto hyperbolic geometry.

Newton gravity ([11] equation 7.1) will be generalized for compatibility with the non-accelerated observer of *Special Relativity* through a "Symmetric Tensor Gravitational Field" ([11] exercise 7.3, with detailed solution in box 7.1).

There, use of a metric tensor $g_{\mu\nu}$ allows to mathematically describe an invariant property $d\tau^2$ as a function of progression in space $d\vec{x}$ and time $dt \equiv dx_0$ of a body as:

$$d\tau^2 = \sum_{\mu,\nu=0}^{3} g_{\mu\nu} dx_\mu dx_\nu \tag{44}$$

Expressing the metric tensor $g_{\mu\nu}$ in terms of $\eta_{\mu\nu}$ (40) and a new tensor $h_{\mu\nu}$ as

$$g_{\mu\nu} := \eta_{\mu\nu} + h_{\mu\nu} \tag{45}$$

the generalization of Newton gravity becomes [7]:

$$\Box h_{\mu\nu} := \left(\partial_0^2 - \sum_{i=1}^{3} \partial_i^2 \right) h_{\mu\nu} = -8\pi M_{\mu\nu} \tag{46}$$

Relations of this form can be solved through so-called "retarded potentials": Whereas the quantities $h_{\mu\nu}$ and $M_{\mu\nu}$ here are a function of space $\vec{x}$ and time $t \equiv x_0$, the solutions for the $h_{\mu\nu}(t, \vec{x})$ can be written in terms of $M_{\mu\nu}(t - |\vec{r} - \vec{x}|, \vec{r})$, i.e. at an earlier time coordinate (see e.g. [11] equation 18.14, or [10] equation 27.16):

$$h_{\mu\nu}(t, \vec{x}) = -2 \int d^3r \frac{M_{\mu\nu}(t - |\vec{r} - \vec{x}|, \vec{x})}{|\vec{r} - \vec{x}|} \tag{47}$$

4.2.5 Linearized Field Equations from General Relativity and "Bootstrap" Argumentation

Gravity is generated by all forms of energy alike, and the resulting force is always attractive. This makes it conceptually simple, however, creates a computational challenge: Because physical fields are also carrier of energy, together with the very masses that generate these fields, one cannot separate the field generating charges (the masses) from the resulting field anymore (as it is possible e.g. with electromagnetism). The gravitational field and its sources are in dynamic balance.

This balance may be viewed from two conceptual angles: 1) "Dynamic geometry is the 'master field' of physics" (from [8] box 18.1 which compares both viewpoints), or 2) a result of "a systematic approximation procedure" (from [10] section 27.6) that starts on a so-called "flat" (non-dynamic) space-time geometry and the "linearized field equations" (as will be discussed now). Immediate "Einstein derivation" of the field equations in *General Relativity* uses the first conceptual viewpoint, and derivations from a flat space-time (like the so-called "spin-2 derivation") the second. Both are proven equivalent and lead to the same effective force [8].

The second approach is sometimes also called the "bootstrap" process due to the iterative nature of its refinement steps. From the "dynamic geometry" viewpoint, this bootstrap process starts at a low energy density approximation (the linearized field equations) and subsequently refines it by consistently taking effects from higher mass densities and velocities into account.

Adopting this argumentation similar to current use e.g. in the "spin-2 model", it will be concluded that it is sufficient for a valid theory of gravitation to show that the immediate source terms of gravitation (masses in motion) generate a gravitational field that is described by the linearized field equations from *General Relativity.*

With a metric tensor $g_{\mu\nu}$ as in (45) $g_{\mu\nu} := \eta_{\mu\nu} + h_{\mu\nu}$ the linearized field equations (see e.g. [11] equation 18.8b, or [10] equation 27.14) can be written in the following form [9]:

$$\Box h_{\mu\nu} = -16\pi \left(M_{\mu\nu} - \frac{1}{2}\rho\eta_{\mu\nu} \right) \tag{48}$$

With use of (43) this becomes

$$\Box h_{\mu\nu} = -8\pi\rho \left(\Lambda^2_{\mathrm{hyp8}} \right)_{\mu\nu} \tag{49}$$

4.3 Proof of Equivalence of the NatAliE Equations Program with the Linearized Field Equations from General Relativity

Lemma 6 *The NatAliE Equations Program for any Mass Density Distribution in Arbitrary Motion is Equivalent to the Linearized Field Equations from General Relativity.*

PROOF. The following will be executed:

[7] From [11] box 7.1 equation 7; note the additional factor 2 from definition of the $\bar{h}$ in equation 7.8c. Also, the energy momentum tensor $T_{\mu\nu}$ is reduced here to the mass tensor $M_{\mu\nu}$ since pressure terms are set to 0.

[8] While proven equivalent, derivations that originate on a flat space-time are sometimes considered inferior to dynamic geometrical derivations, since any such flat space-time is proven to be not observable, and therefore out of reach by the experiment. It has even been shown that one must necessarily arrive at the *General Relativity* field equations even when starting from an (unobservable) "arbitrary a-priory background" [12], which does not need to be flat. Investigation of the invariant hypernumber modulus here may overcome this lack of experimental accessibility, by demonstrating a fundamental mathematical arithmetic suitable for joint description of a new quantum gravity and traditional quantum mechanics.

[9] Note that per definition the $\bar{h}$ in [11] or $\bar{f}$ in [10] consist of the $h_{\mu\nu}$ used here and an additional effective term $-8\pi \left[\sum_{\rho,\sigma=0}^{3} T_{\rho\sigma}\eta_{\rho\sigma} \right] \eta_{\mu\nu}$ which reduces for our choice of internal pressure $p = 0$ to $-8\pi\rho\eta_{\mu\nu}$. This term does not vary as function of space and time, and is written explicitely here on the right side of the equation.

(1) Equation (46) is a relation that separates the generators of gravity $M_{\mu\nu}$ from their effect on otherwise hyperbolic space-time $\Box h_{\mu\nu}$.
(2) In its solutions (47) the terms ρ and $1/|\vec{r}-\vec{x}|$ can be identified as purely gravitational: ρ is the mass distribution that generates gravitation, and $|\vec{r}-\vec{x}|$ is the effective distance for the gravitational force. Both terms ρ and $1/|\vec{r}-\vec{x}|$ must be replaced with their projected values ρ' and $1/|\vec{r}'-\vec{x}'|$ from circular (Euclidean) geometry.
(3) Since (46) is a linear differential equation, one can calculate the effective solution for any ρ' distribution by linear superposition of its individual constituents.
(4) For a single constituent mass $m_{\mathrm{cir8}(l)}$ the coordinates will be transformed in a way that allows to calculate the effect of length expansion from (28) in the direction of motion $\vec{v}_{(l)} = d\vec{x}_{(l)}/dt_{(l)}$ of the constituent $m_{\mathrm{cir8}(l)}$.
(5) The result for this special choice of coordinates will be generalized in tensor form, to be valid in any coordinate system.
(6) Since the resulting relation is linear and in tensor form, one is able to generalize the result for any number of individual constituents $m_{\mathrm{hyp8}(l)}$.
(7) In its generalized tensor form, the relation is equivalent to the linearized field equations from *General Relativity* in their low energy density approximation assumption.

This will conclude the proof.

4.3.1 Gravity on Hyperbolic Space-time

As shown in great detail in [11] exercise 7.3 and its solution in box 7.1, relation (46) defines a gravitation consistent with *Special Relativity*, which was proven above to be described through the hyperbolic subalgebra of the invariant conic sedenion modulus. This relation therefore qualifies as equation of motion on hyperbolic space-time geometry, onto which gravitational effects from circular geometry can be projected according to the *NatAliE* equations program.

Terms $h_{\mu\nu}$ and $M_{\mu\nu}$ will now be replaced with $h'_{\mu\nu}$ and $M'_{\mu\nu}$ to symbolize that they are subject to modifications due to projected results from circular geometry:

$$\Box h'_{\mu\nu} = -8\pi M'_{\mu\nu} \tag{50}$$

4.3.2 Identifying Gravitational and Non-Gravitational Terms

The solutions of (50) are similar to (47) and allow to identify terms that originate from the gravitational force: Masses $M'_{\mu\nu}$ are the generators of a force that decreases with distance $|\vec{r}'-\vec{x}'|$:

$$h'_{\mu\nu}(t,\vec{x}) = -2\int d^3r \frac{M'_{\mu\nu}(t-|\vec{r}-\vec{x}|,\vec{x})}{|\vec{r}'-\vec{x}'|} \tag{51}$$

Location and motion of the charges, on the other hand, will continue to be expressed in coordinates t and $\vec{x}$ that are consistent with *Special Relativity*. This projects the influence of a gravitational force from circular space-time onto an equation of motion on hyperbolic space-time.

4.3.3 Projected Results from Circular Geometry on a Single Constituent of $M_{\mu\nu}$

The general mass distribution ρ' will now be separated into distinct constituent masses $m_{\mathrm{cir8}(l)}$:

$$\rho' := \sum_l m_{\mathrm{cir8}(l)} \tag{52}$$

Since (50) is linear in $h'_{\mu\nu}$ and $M'_{\mu\nu} := \rho' u_\mu u_\nu$ it is possible to calculate the effective result of any mass distribution ρ' with individual four-velocities u_μ by linear superposition.

4.3.4 Special Choice of Coordinates for a Single $m_{\mathrm{cir8}(l)}$

For a single $m_{\mathrm{cir8}(l)}$ coordinates will be chosen for which $u_{(l)\mu} = (1,0,0,0)$ and $m_{\mathrm{cir8}(l)}$ is located at $\vec{r}'_{(l)} = (0,0,0)$. This describes a body with mass $m_{\mathrm{cir8}(l)}$ that is at rest in the coordinate origin. The mass tensor $M'_{(l)\mu\nu}$ (38) is then reduced to

$$M'_{(l)\mu\nu} = \begin{cases} m_{\mathrm{cir8}(l)} & (\mu=\nu=0) \\ 0 & \text{otherwise} \end{cases} \tag{53}$$

and (51) only yields a non-zero result in $h'_{(l)\mu\nu}$ for $\mu=\nu=0$:

$$h'_{(l)00}(t,\vec{x}) = -2\frac{m_{\mathrm{cir8}(l)}\left(t-|\vec{x}|,0\right)}{|\vec{x}'|} \tag{54}$$

In this choice of coordinates, the relative speed of any observer [10] $\vec{v}_{(l)}$ with respect to $m_{\mathrm{cir8}(l)}$ is then simply:

$$\vec{v}_{(l)} = \frac{d\vec{x}}{dt} \tag{55}$$

According to (27) the projected value of $m_{\mathrm{cir8}(l)}$ from circular onto hyperbolic space-time is the effective mass that generates gravitational force on this observer:

$$m_{\mathrm{cir8}(l)} = m_{\mathrm{hyp8}(l)}\sqrt{\frac{1+\left|\vec{v}_{(l)}\right|^2}{1-\left|\vec{v}_{(l)}\right|^2}} \tag{56}$$

Also, the distance effective for the gravitational force is length expanded (28):

$$|\vec{x}'| = |\vec{x}|\sqrt{1+\left|\vec{v}_{(l)}\right|^2} \tag{57}$$

This yields $h'_{(l)00}(t,\vec{x})$ in terms of the effective projected values as:

$$h'_{(l)00}(t,\vec{x}) = -2\frac{m_{\mathrm{hyp8}(l)}\left(t-|\vec{x}|,0\right)}{|\vec{x}|}\frac{1+\left|\vec{v}_{(l)}\right|^2}{\sqrt{1-\left|\vec{v}_{(l)}\right|^2}} \tag{58}$$

With definition of a field

$$\gamma_{(l)} := \left(1-\left|\vec{v}_{(l)}\right|^2\right)^{-1/2} \tag{59}$$

and the special choice of coordinates that put the progression of an observer in space $d\vec{x}$ and time dt into the direction of relative speed $\vec{v} = d\vec{x}/dt$ between observer and $m_{\mathrm{hyp8}(l)}$, one can identify the effective metric from (44):

$$d\tau^2 = \left(1+h'_{(l)00}\left(t,\vec{x}\right)\right)dt^2 - |d\vec{x}|^2 \tag{60}$$

$$= \left(1-2\gamma_{(l)}\frac{m_{\mathrm{hyp8}(l)}\left(t-|\vec{x}|,0\right)}{|\vec{x}|}\left(1+\left|\frac{d\vec{x}}{dt}\right|^2\right)\right)dt^2 - |d\vec{x}|^2 \tag{61}$$

$$= \left(1-2\gamma_{(l)}\frac{m_{\mathrm{hyp8}(l)}\left(t-|\vec{x}|,0\right)}{|\vec{x}|}\right)dt^2 - \left(1+2\gamma_{(l)}\frac{m_{\mathrm{hyp8}(l)}\left(t-|\vec{x}|,0\right)}{|\vec{x}|}\right)|d\vec{x}|^2 \tag{62}$$

This relation (62) can then be interpreted as space-time metric that is generated from an effective $h_{(l)\mu\nu}(t,\vec{x})$:

$$h_{(l)\mu\nu}(t,\vec{x}) := -2\gamma_{(l)}\frac{m_{\mathrm{hyp8}(l)}\left(t-|\vec{x}|,0\right)}{|\vec{x}|}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} \tag{63}$$

$$= -2\gamma_{(l)}\frac{m_{\mathrm{hyp8}(l)}\left(t-|\vec{x}|,0\right)}{|\vec{x}|}\delta_{\mu\nu} \tag{64}$$

[10] An observer is any object that experiences the body's gravitational force; coordinates are chosen individually for each $m_{\mathrm{cir8}(l)}$ so that the observer is in motion, whereas $m_{\mathrm{cir8}(l)}$ is at rest. This accounts for the locality of the circular Lorentz transformation (22).

4.3.5 Generalization for Any Frame of Reference

Relation (64) can be interpreted as solution to a new linear differential equation

$$\Box h_{(l)\mu\nu}(t,\vec{x}) = -8\pi\gamma_{(l)} m_{\mathrm{hyp8}(l)}\,(t,0)\,\delta_{\mu\nu} \tag{65}$$

and the Kronecker symbol $\delta_{\mu\nu}$ can be generalized to the tensor $\Lambda^2_{\mathrm{hyp8}}$ (32) for any choice of coordinates:

$$\Box h_{(l)\mu\nu} = -8\pi\gamma_{(l)} m_{\mathrm{hyp8}(l)} \Lambda^2_{\mathrm{hyp8}} \tag{66}$$

Using (43) this can be written as:

$$\Box h_{(l)\mu\nu} = -16\pi\gamma_{(l)} \left(M_{(l)\mu\nu} - \frac{1}{2} m_{\mathrm{hyp8}(l)} \eta_{\mu\nu} \right) \tag{67}$$

4.3.6 Generalization for Any Mass Distribution

In analogy to the argumentation that leads from (42) to (43), all individual contributions from all $m_{\mathrm{hyp8}(l)}$ terms can be added subsequently: The linear relations (66) and (67) in tensor form are valid for any equivalent frame of reference, and individual contributions from the $m_{\mathrm{hyp8}(l)}$ can be added up through linear superposition of the respective results.

With

$$\rho := \sum_l m_{\mathrm{hyp8}(l)} \tag{68}$$

relation (67) becomes for any mass distribution $M_{\mu\nu}$:

$$\Box h_{\mu\nu} = -16\pi\gamma \left(M_{\mu\nu} - \frac{1}{2} \rho \eta_{\mu\nu} \right) \tag{69}$$

4.3.7 Equivalence with Linearized Field Equations from General Relativity

The relation (69) as derived from the *NatAliE* equations program only differs from (48) by a velocity field $\gamma = (1 - |\vec{v}|^2)^{-1/2}$. In the low energy density approximation used to derive the linearized field equations from *General Relativity*, only lowest order occurrences of such factors γ are carried forward. Additional factors are are approximated $\gamma \approx 1$ and are considered a higher-order correction.

From the "bootstrap" argumentation it is possible to quantify the exact magnitude of such higher-order corrections. Generally, a significant increase in velocities in the source field $M_{\mu\nu}$ leads to stronger curvature $\Box h_{\mu\nu}$. An additional field factor γ in (69) has at least approximately the same effect, and can therefore be considered a higher-order correction that is excluded in the derivation approach of (48).

Therefore, the field equations as derived from the *NatAliE* equations program are equivalent to the linearized field equations as obtained from *General Relativity*, which proves lemma 6.

5 Conclusion, Outlook, and Comparison with Other Approaches to Physics on Non-traditional Numbers

5.1 Hypernumber Modulus Invariance Theorem Conclusion

The fundamental principles that govern *Special Relativity* were shown to be correctly represented through the hyperbolic octonion subalgebra of the invariant conic sedenion modulus (7). The corresponding circular octonion subalgebra was used in the argument for proposition 4, for the existence of a *NatAliE* equations program that correctly describes large body gravitation. It was argued that lack of empirical information on quantum gravitational behavior, or a generally accepted mathematical description thereof, prohibits conclusive evaluation of this proposition at the moment: Hypernumber relations in this paper were rationalized from comparison with current description of quantum mechanics, whereas *General Relativity* can only be confirmed for non-quantum gravitation. It was argued accordingly that the *NatAliE* equations program is justifiable with respect to the current state of mathematical description of physical law, and compatibility with gravity from *General Relativity* was shown.

Therefore, it is now concluded that proper foundation for the hypernumber modulus invariance theorem (theorem 1) has been established, limited only by outstanding experimental validation of its underlying assumptions.

5.2 Outlook

Many concepts from physics were introduced and referred to when investigating the *NatAliE* equations program. With $|d\tau_{\text{con16}}|$ as established invariance condition that warrants validity of physical law in any equivalent frame of reference, these concepts are not needed anymore: Further investigation of $|d\tau_{\text{con16}}|$ may focus only on geometrical and number aspects, and physical quantities (like E, $\vec{p}$, or m) may be treated (again) without having to attach further meaning or interpretation from physics; they may remain mere constants or Fourier coefficients.

In order to qualify conic sedenion arithmetic as a complete description of quantum mechanics, with respect to both the electromagnetic and gravitational force, the concept of a physical "force field" is needed at last. The force field concept realizes mathematical description of dynamic interaction between its generating charges, as well as field self-interaction. Such a concept needs to be understood in terms of hypernumber geometry, to possibly find support for some argumentation from physics used above (e.g. the gravitational force field is known to be a carrier of energy and therefore is generator of gravity itself).

5.3 Comparison with Other Approaches to Physics on Non-traditional Numbers

Charles Musès (1919-2000), who invented hypernumber arithmetics, has over time proposed potential approaches to physics on hypernumbers. The following provides reference to some of Musès' ideas in this field and distinguishes the current approach. In addition, other recent investigation of physics on "split-octonions" is mentioned.

Investigation into quantum gravity in [13] explores similarity between hypernumber arithmetics and relations that appear on "spin 2" gravity models in quantum field theory. These models are built on traditional hyperbolic Minkowski space-time only, and are subject to further investigation whether or not projection of a gravitational exchange particle on circular space-time would become a spin 2 particle in hyperbolic metric, and how such projection could be rationalized.

In the current paper, space and time are treated as coordinates with identical properties; they are assigned to different coefficients in conic sedenion representation (3). This is in contrast to the notion in [14] which criticizes this approach (e.g. section "The Asymmetry of Time and Space"). However, if one speculatively interprets the invariant hypernumber modulus dT (7) as some kind of "universal time" that always increases, this interpretation could become compatible with Musès' argumentation, due to the modularity of hypernumber bases: Being a modulus, dT represents pure magnitude without direction, and is therefore distinct from the dimensionality of number system components. While some space and time dimensions could still be associated with individual number system coefficients, this new universal time would pose a dimensionless unifying concept.

A physical model that uses an unobservable extra dimension is proposed in [15]. While it is not obvious in this model how relativity (as discussed in the current paper) could be warranted e.g. with respect to constant speed of electromagnetic radiation (light), it is interesting to potentially tie unobservable physical dimensions to w hypernumbers, as suggested in the closing note there.

In [16] Musès demonstrates how conic sedenions support a generalized concept of "reflection" as continuous geometrical transformation from one into a mirrored state. Together with other important properties of hypernumbers (e.g. the product modulus law, and certain power orbit properties), these reflections and similar transformations could become applicable in physics: Interpreted as symmetry transformations on wave functions they may be further examined with respect to description of fundamental forces.

Recent investigation by Merab Gogberashvili on so-called "split-octonions" ([17,18]) is also expressing fundamental physical relations (the Dirac equation with electromagnetic field) in a number system that is not simply a matrix extension of traditional complex numbers. It would be interesting to examine split-octonions and hypernumber octonion types for potential correlation.

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