Signature of gravity in conic sedenions

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Abstract

A method was demonstrated earlier on how extension of complex number algebra using nonreal square roots of +1 could potentially aid mathematical description of physical law, by transitioning between different geometries through genuine hypernumber rotation. The Dirac equation in physics can be expressed on hyperbolic octonion algebra and then transformed into a counterpart on circular octonions, by means of conic sedenions as unifying number concept. This paper examines potential applicability of this approach by calculating eigenvectors and Green's function of the circular octonion counterpart to the classical Dirac equation. The results exhibit behavior that one might expect from a quantum gravitational primitive.

Key words: hypernumbers; conic complex numbers; M-algebra; sedenions; octonions; physics on hyperbolic and circular geometry; quantum gravity

1 Introduction

Motion of free spin 1/2 particles (like e.g. electrons or protons) is described in quantum physics by the Dirac equation. This fundamental building block in current description of electromagnetic, weak, and strong forces in nature has been shown to be expressible through octonionic algebras [1,2]. A method was demonstrated in [1] to transition the Dirac equation between different geometries using genuine hypernumber rotation [1]. In this paper, eigenvectors and Green's function of a circular octonionic counterpart to the classical Dirac equation, obtained after rotation in the $(1, i_0)$ plane, will yield behavior that one might require from a relation that is fundamental to quantum gravity.

While the hypernumber relation under investigation will be mapped onto traditional matrix and vector form on circular complex numbers, it must be emphasized that this procedure is only chosen here for practical reasons: to use traditional mathematical tools in examining a new and speculative concept. If physical law would indeed further materialize through use of hypernumber arithmetics, more genuine application of the respective number systems will be indicated.

2 Circular and Hyperbolic Dirac Equations

The conic sedenion relation

$$\bigtriangledown_{\text{con16}} \Psi_{\text{con16}} = 0 \tag{1}$$

to basis elements $b_{\text{con16}} \in \{1, i_1, ..., i_7, i_0, \varepsilon_1, ..., \varepsilon_7\}$ can be transitioned from circular to hyperbolic geometry [1] using a real number coefficient α

$$\bigtriangledown_{\text{con16}} := \bigtriangledown_{\text{Q1}} + \exp(\alpha i_0) \bigtriangledown_{\text{Q2}} \tag{2}$$

$$\Psi_{\text{con16}} := \Psi_{\text{Q1}} + \exp(\alpha i_0) \Psi_{\text{Q2}} \tag{3}$$

and the following definitions [2]:

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[1] Notation and definitions will be carried forward from there; for detailed analysis of the pertaining hypernumber systems see [3,4].

[2] Please note that definition (4) in [1] is incorrect and should be $\nabla_{\text{hyp8}} := (-m, \partial_0, 0, 0, 0, -\partial_3, \partial_2, -\partial_1)$, as well as definition (3) should be $\Psi_{\text{hyp8}} := \left(\psi_0^{\text{r}}, \psi_0^{\text{i}}, \psi_1^{\text{r}}, \psi_1^{\text{i}}, \psi_2^{\text{r}}, -\psi_2^{\text{i}}, -\psi_3^{\text{r}}, -\psi_3^{\text{i}}\right)$. The different definitions were the result of using a conic sedenion multiplication table which identified the classical octonion element "l" with sedenion element $-i_4$ instead of i_4, therefore not being consistent with the cited sources [3,6].

$$\triangledown_{\mathrm{Q1}} := (-m, \partial_0, 0,0,0,0,0,0, \quad 0,0,0,0,0,0,0,0) \tag{4}$$
$$\triangledown_{\mathrm{Q2}} := (0,0,0,0,0, \partial_3, -\partial_2, \partial_1, \quad 0,0,0,0,0,0,0,0) \tag{5}$$
$$\Psi_{\mathrm{Q1}} := \left(\psi_0^{\mathrm{r}}, \psi_0^{\mathrm{i}}, \psi_1^{\mathrm{r}}, \psi_1^{\mathrm{i}}, 0,0,0,0, \quad 0,0,0,0,0,0,0,0\right) \tag{6}$$
$$\Psi_{\mathrm{Q2}} := \left(0,0,0,0, -\psi_2^{\mathrm{r}}, \psi_2^{\mathrm{i}}, \psi_3^{\mathrm{r}}, \psi_3^{\mathrm{i}}, \quad 0,0,0,0,0,0,0,0\right) \tag{7}$$

The classical ("hyperbolic") Dirac equation then corresponds to $\alpha = \pi/2$ and a new counterpart on circular geometry to $\alpha = 0$.

The latter ($\alpha = 0$) is subject to investigation in this paper and will be called "circular" Dirac equation. The conic sedenion relation reduces in this case to its circular octonion subalgebra z_{cir8} to basis elements $b_{\mathrm{cir8}} \in \{1, i_1, ..., i_7\}$ in the mapping:

$$(c[1], c[i_1], c[i_2], c[i_3], c[i_4], c[i_5], c[i_6], c[i_7], \quad 0,0,0,0,0,0,0,0) \mapsto z_{\mathrm{cir8}} \tag{8}$$

The relation

$$\triangledown_{\mathrm{cir8}} \Psi_{\mathrm{cir8}} = 0 \tag{9}$$

with $\triangledown_{\mathrm{Q1}} + \triangledown_{\mathrm{Q2}} \mapsto \triangledown_{\mathrm{cir8}}$ and $\Psi_{\mathrm{Q1}} + \Psi_{\mathrm{Q2}} \mapsto \Psi_{\mathrm{cir8}}$ can also be written in traditional matrix form on circular complex numbers as:

$$\begin{pmatrix} -m + i\partial_0 & 0 & i\partial_3 & i\partial_1 + \partial_2 \\ 0 & -m + i\partial_0 & i\partial_1 - \partial_2 & -i\partial_3 \\ i\partial_3 & i\partial_1 + \partial_2 & -m - i\partial_0 & 0 \\ i\partial_1 - \partial_2 & -i\partial_3 & 0 & -m - i\partial_0 \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{10}$$

(with $\psi_\mu := \psi_\mu^{\mathrm{r}} + i\psi_\mu^{\mathrm{i}}$, $\mu \in \{0,1,2,3\}$).

Proof: The previous statement will be validated in analogy to the classical Dirac equation mapping in [1]. The four relations of (10) are separated into their real and imaginary parts:

$$(-m + i\partial_0)\left(\psi_0^{\mathrm{r}} + i\psi_0^{\mathrm{i}}\right) + i\partial_3\left(\psi_2^{\mathrm{r}} + i\psi_2^{\mathrm{i}}\right) + (i\partial_1 + \partial_2)\left(\psi_3^{\mathrm{r}} + i\psi_3^{\mathrm{i}}\right) = 0$$
$$-m\psi_0^{\mathrm{r}} - \partial_0\psi_0^{\mathrm{i}} - \partial_1\psi_3^{\mathrm{i}} + \partial_2\psi_3^{\mathrm{r}} - \partial_3\psi_2^{\mathrm{i}} = 0 \tag{11}$$
$$-m\psi_0^{\mathrm{i}} + \partial_0\psi_0^{\mathrm{r}} + \partial_1\psi_3^{\mathrm{r}} + \partial_2\psi_3^{\mathrm{i}} + \partial_3\psi_2^{\mathrm{r}} = 0 \tag{12}$$

$$(-m + i\partial_0)\left(\psi_1^{\mathrm{r}} + i\psi_1^{\mathrm{i}}\right) + (i\partial_1 - \partial_2)\left(\psi_2^{\mathrm{r}} + i\psi_2^{\mathrm{i}}\right) - i\partial_3\left(\psi_3^{\mathrm{r}} + i\psi_3^{\mathrm{i}}\right) = 0$$
$$-m\psi_1^{\mathrm{r}} - \partial_0\psi_1^{\mathrm{i}} - \partial_1\psi_2^{\mathrm{i}} - \partial_2\psi_2^{\mathrm{r}} + \partial_3\psi_3^{\mathrm{i}} = 0 \tag{13}$$
$$-m\psi_1^{\mathrm{i}} + \partial_0\psi_1^{\mathrm{r}} + \partial_1\psi_2^{\mathrm{r}} - \partial_2\psi_2^{\mathrm{i}} - \partial_3\psi_3^{\mathrm{r}} = 0 \tag{14}$$

$$i\partial_3\left(\psi_0^{\mathrm{r}} + i\psi_0^{\mathrm{i}}\right) + (i\partial_1 + \partial_2)\left(\psi_1^{\mathrm{r}} + i\psi_1^{\mathrm{i}}\right) - (m + i\partial_0)\left(\psi_2^{\mathrm{r}} + i\psi_2^{\mathrm{i}}\right) = 0$$
$$-m\psi_2^{\mathrm{r}} + \partial_0\psi_2^{\mathrm{i}} - \partial_1\psi_1^{\mathrm{i}} + \partial_2\psi_1^{\mathrm{r}} - \partial_3\psi_0^{\mathrm{i}} = 0 \tag{15}$$
$$-m\psi_2^{\mathrm{i}} - \partial_0\psi_2^{\mathrm{r}} + \partial_1\psi_1^{\mathrm{r}} + \partial_2\psi_1^{\mathrm{i}} + \partial_3\psi_0^{\mathrm{r}} = 0 \tag{16}$$

$$(i\partial_1 - \partial_2)\left(\psi_0^{\mathrm{r}} + i\psi_0^{\mathrm{i}}\right) - i\partial_3\left(\psi_1^{\mathrm{r}} + i\psi_1^{\mathrm{i}}\right) - (m + i\partial_0)\left(\psi_3^{\mathrm{r}} + i\psi_3^{\mathrm{i}}\right) = 0$$
$$-m\psi_3^{\mathrm{r}} + \partial_0\psi_3^{\mathrm{i}} - \partial_1\psi_0^{\mathrm{i}} - \partial_2\psi_0^{\mathrm{r}} + \partial_3\psi_1^{\mathrm{i}} = 0 \tag{17}$$
$$-m\psi_3^{\mathrm{i}} - \partial_0\psi_3^{\mathrm{r}} + \partial_1\psi_0^{\mathrm{r}} - \partial_2\psi_0^{\mathrm{i}} - \partial_3\psi_1^{\mathrm{r}} = 0 \tag{18}$$

The circular octonion product $\triangledown_{\mathrm{cir8}} \Psi_{\mathrm{cir8}}$ is explicitely:

$$\begin{aligned} & (-m, \partial_0, 0, 0, \quad 0, \partial_3, -\partial_2, \partial_1)\left(\psi_0^{\mathrm{r}}, \psi_0^{\mathrm{i}}, \psi_1^{\mathrm{r}}, \psi_1^{\mathrm{i}}, \quad -\psi_2^{\mathrm{r}}, \psi_2^{\mathrm{i}}, \psi_3^{\mathrm{r}}, \psi_3^{\mathrm{i}}\right) \\ = \quad & (-m\psi_0^{\mathrm{r}},\ -m\psi_0^{\mathrm{i}},\ -m\psi_1^{\mathrm{r}},\ -m\psi_1^{\mathrm{i}}, \quad m\psi_2^{\mathrm{r}},\ -m\psi_2^{\mathrm{i}},\ -m\psi_3^{\mathrm{r}},\ -m\psi_3^{\mathrm{i}}) \\ & +(-\partial_0\psi_0^{\mathrm{i}},\ \partial_0\psi_0^{\mathrm{r}},\ -\partial_0\psi_1^{\mathrm{i}},\ \partial_0\psi_1^{\mathrm{r}},\ -\partial_0\psi_2^{\mathrm{i}},\ -\partial_0\psi_2^{\mathrm{r}},\ \partial_0\psi_3^{\mathrm{i}},\ -\partial_0\psi_3^{\mathrm{r}}) \\ & +(-\partial_3\psi_2^{\mathrm{i}},\ \partial_3\psi_2^{\mathrm{r}},\ \partial_3\psi_3^{\mathrm{i}},\ -\partial_3\psi_3^{\mathrm{r}},\ \partial_3\psi_0^{\mathrm{i}},\ \partial_3\psi_0^{\mathrm{r}},\ \partial_3\psi_1^{\mathrm{i}},\ -\partial_3\psi_1^{\mathrm{r}}) \\ & +(\ \partial_2\psi_3^{\mathrm{r}},\ \partial_2\psi_3^{\mathrm{i}},\ -\partial_2\psi_2^{\mathrm{r}},\ -\partial_2\psi_2^{\mathrm{i}},\ -\partial_2\psi_1^{\mathrm{r}},\ \partial_2\psi_1^{\mathrm{i}},\ -\partial_2\psi_0^{\mathrm{r}},\ -\partial_2\psi_0^{\mathrm{i}}) \\ & +(-\partial_1\psi_3^{\mathrm{i}},\ \partial_1\psi_3^{\mathrm{r}},\ -\partial_1\psi_2^{\mathrm{i}},\ \partial_1\psi_2^{\mathrm{r}},\ \partial_1\psi_1^{\mathrm{i}},\ \partial_1\psi_1^{\mathrm{r}},\ -\partial_1\psi_0^{\mathrm{i}},\ \partial_1\psi_0^{\mathrm{r}}) \end{aligned}$$

$$
\begin{aligned}
= (\ & -m\psi_0^{\mathrm{r}} - \partial_0\psi_0^{\mathrm{i}} - \partial_3\psi_2^{\mathrm{i}} + \partial_2\psi_3^{\mathrm{r}} - \partial_1\psi_3^{\mathrm{i}}\ , \\
& -m\psi_0^{\mathrm{i}} + \partial_0\psi_0^{\mathrm{r}} + \partial_3\psi_2^{\mathrm{r}} + \partial_2\psi_3^{\mathrm{i}} + \partial_1\psi_3^{\mathrm{r}}\ , \\
& -m\psi_1^{\mathrm{r}} - \partial_0\psi_1^{\mathrm{i}} + \partial_3\psi_3^{\mathrm{i}} - \partial_2\psi_2^{\mathrm{r}} - \partial_1\psi_2^{\mathrm{i}}\ , \\
& -m\psi_1^{\mathrm{i}} + \partial_0\psi_1^{\mathrm{r}} - \partial_3\psi_3^{\mathrm{r}} - \partial_2\psi_2^{\mathrm{i}} + \partial_1\psi_2^{\mathrm{r}}\ , \\
& \ \ m\psi_2^{\mathrm{r}} - \partial_0\psi_2^{\mathrm{i}} + \partial_3\psi_0^{\mathrm{i}} - \partial_2\psi_1^{\mathrm{r}} + \partial_1\psi_1^{\mathrm{i}}\ , \\
& -m\psi_2^{\mathrm{i}} - \partial_0\psi_2^{\mathrm{r}} + \partial_3\psi_0^{\mathrm{r}} + \partial_2\psi_1^{\mathrm{i}} + \partial_1\psi_1^{\mathrm{r}}\ , \\
& -m\psi_3^{\mathrm{r}} + \partial_0\psi_3^{\mathrm{i}} + \partial_3\psi_1^{\mathrm{i}} - \partial_2\psi_0^{\mathrm{r}} - \partial_1\psi_0^{\mathrm{i}}\ , \\
& -m\psi_3^{\mathrm{i}} - \partial_0\psi_3^{\mathrm{r}} - \partial_3\psi_1^{\mathrm{r}} - \partial_2\psi_0^{\mathrm{i}} + \partial_1\psi_0^{\mathrm{r}}\ )
\end{aligned}
\tag{19}
$$

The eight components of (19) can be identified as left-hand part of equations (11) through (18), either identical or with the opposite sign. This proves that $\bigtriangledown_{\mathrm{cir8}}\Psi_{\mathrm{cir8}} = 0$ (9) is indeed equivalent to the circular Dirac equation in matrix form on circular complex numbers.

3 Eigenvectors of the Circular Dirac Equation

The constants $\vec{p} := (p_1, p_2, p_3)$ and $E := p_0$ will be introduced, together with space $\vec{x} := (x_1, x_2, x_3)$ and time $t := x_0$. This allows to specify four linear independent solutions of (10). If interpreted as eigenvalue equation, these solutions are eigenvectors to the eigenvalue m:

$$
\Psi_1^+ := \exp i\left(\vec{p}\vec{x} - Et\right)\begin{pmatrix} 1 \\ 0 \\ -p_3/\left(m+E\right) \\ \left(-p_1 - ip_2\right)/\left(m+E\right) \end{pmatrix} \tag{20}
$$

$$
\Psi_1^- := \exp i\left(\vec{p}\vec{x} - Et\right)\begin{pmatrix} 0 \\ 1 \\ \left(-p_1 + ip_2\right)/\left(m+E\right) \\ p_3/\left(m+E\right) \end{pmatrix} \tag{21}
$$

$$
\Psi_2^+ := \exp i\left(\vec{p}\vec{x} + Et\right)\begin{pmatrix} -p_3/\left(m+E\right) \\ \left(-p_1 - ip_2\right)/\left(m+E\right) \\ 1 \\ 0 \end{pmatrix} \tag{22}
$$

$$
\Psi_2^- := \exp i\left(\vec{p}\vec{x} + Et\right)\begin{pmatrix} \left(-p_1 + ip_2\right)/\left(m+E\right) \\ p_3/\left(m+E\right) \\ 0 \\ 1 \end{pmatrix} \tag{23}
$$

The solutions $\Psi_1^{\pm}$ only differ from the $\Psi_2^{\pm}$ through the sign before Et in the exponent, aside from simple reordering of vector and matrix indices $(0,1,2,3) \rightarrow (2,3,0,1)$. This is in contrast to the analogous solutions of the classical Dirac equation in physics, where additional changes in the vector part of the eigenvectors accompany the difference in exponent sign [3] .

[3] See introductions into Quantum Electrodynamics like e.g. [5], equation (23.11). There, the particle solution ψ_p (23.1) for positive

Carrying forward this analogy to classical physics and interpreting the $\Psi_1^\pm$ and $\Psi_2^\pm$ as particle and anti-particle solutions respectively, one finds that in the circular case the Dirac equation only distinguishes between particles $(-Et)$ and anti-particles $(+Et)$ through our choice of time axis. This invariance of the free spin 1/2 particle solutions under time reversal may be an expected property of a formalism describing gravitational force, which affects both particle types identically.

4 Green's Function of the Circular Dirac Equation

Independent from the actual method of introducing a gravitational field into the circular Dirac equation, one may expect that its Green's function will also remain invariant under time reversal. In physics, the Green's function on the classical hyperbolic Dirac equation is called the particle's "propagator" (e.g. [5] §75).

With use of

$$\beta_0 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \beta_2 := \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \tag{24}$$

$$\beta_1 := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad \beta_3 := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$\Psi := \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \qquad m \equiv \begin{pmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & m \end{pmatrix} \tag{25}$$

the circular Dirac equation (10) can be written abbreviated as [4]:

$$\left[i \sum_{\nu=0}^{3} \beta_\nu \partial_\nu - m \right] \Psi = 0 \tag{26}$$

Its Green's function $G(x-y)$ with four-vectors x and y (y constant) then satisfies:

$$\left[i \sum_{\nu=0}^{3} \beta_\nu \partial_\nu - m \right] G(x-y) = \delta^4(x-y) \tag{27}$$

Expressing both $G(x-y)$ and $\delta^4(x-y)$ as Fourier transforms

$$G(x-y) = \int \frac{d^4p}{(2\pi)^4} G(p) \exp\left[-i \sum_{\nu=0}^{3} p_\nu (x_\nu - y_\nu) \right] \tag{28}$$

$$\delta^4(x-y) = \int \frac{d^4p}{(2\pi)^4} \exp\left[-i \sum_{\nu=0}^{3} p_\nu (x_\nu - y_\nu) \right] \tag{29}$$

energies ("frequencies") $\varepsilon = +E$ is proportional to $\exp(-ipx)$ which contains a metric tensor (Minkowski metric): $\psi_p \propto \exp(-ipx) = \exp i(\vec{p}\vec{x} - Et)$. Similarly, the anti-particle solution ("negative frequencies"; $\varepsilon = -E$) from (23.2) is: $\psi_{-p} \propto \exp(ipx) = \exp i(-\vec{p}\vec{x} + \varepsilon t) = \exp i(-\vec{p}\vec{x} - Et)$. Therefore, ψ_p exhibits the same space-time dependency as $\Psi_1^\pm$, and ψ_{-p} corresponds to $\Psi_2^\pm$ accordingly.

[4] Please note that all summations will be spelled out, and all summation indices will be lower indices. This is different from notation typical in physics, where the sum over duplicate indices is executed by default, and upper and lower indices indicate the presence of a metric tensor (like Minkowski metric $\eta_{\mu\nu}$). The notation chosen here avoids a potential ambiguity: In the case of circular octonions, the metric is Euclidean and the metric tensor $\delta_{\mu\nu}$ is unity and can be omitted. In the case of the classical hyperbolic Dirac equation, the metric is Minkowski and would be written explicitely as $\eta_{\mu\nu}$ if present.

yields:

$$\left(\sum_{\nu=0}^{3}\beta_\nu p_\nu - m\right) G(p) = 1 \tag{30}$$

This can be solved by using the β matrix summation rule

$$\frac{1}{2}\left(\beta_\mu\beta_\nu + \beta_\nu\beta_\mu\right) = \delta_{\mu\nu} \tag{31}$$

and the identity

$$\begin{aligned}\sum_{\mu=0}^{3}\sum_{\nu=0}^{3}\beta_\mu\beta_\nu p_\mu p_\nu &= \frac{1}{2}\sum_{\mu=0}^{3}\sum_{\nu=0}^{3}\left[\beta_\mu\beta_\nu p_\mu p_\nu + \beta_\nu\beta_\mu p_\nu p_\mu\right]\\ &= \frac{1}{2}\sum_{\mu=0}^{3}\sum_{\nu=0}^{3}\left[\left(\beta_\mu\beta_\nu + \beta_\nu\beta_\mu\right) p_\mu p_\nu\right]\\ &= \sum_{\mu=0}^{3}\sum_{\nu=0}^{3}\delta_{\mu\nu} p_\mu p_\nu = \sum_{\nu=0}^{3} p_\nu^2\end{aligned} \tag{32}$$

to:

$$G(p) = \frac{\sum_{\nu=0}^{3}\beta_\nu p_\nu + m}{\sum_{\nu=0}^{3} p_\nu^2 - m^2} \tag{33}$$

The Green's function $G(p)$ has a pole for all $m^2 = \sum_{\nu=0}^{3} p_\nu^2 = E^2 + |\vec{p}|^2$, which is distinct from classical physics where the propagator's pole is at all $m^2 = \sum_{\mu=0}^{3}\sum_{\nu=0}^{3}\eta_{\mu\nu}p_\mu p_\nu = E^2 - |\vec{p}|^2$ (see e.g. [5] from equation 75.10 and following). There, the difference in sign before E^2 and $|\vec{p}|^2$ (the Fourier coefficients of time x_0 and space $\vec{x}$) requires special consideration with respect to whether events are in the future ($x_0 < y_0$) or in the past ($x_0 > y_0$), and treatment of the poles distinguishes particle and anti-particle behavior.

For the circular Dirac equation, however, the pole at $m^2 = E^2 + |\vec{p}|^2$ reflects the Euclidean geometry of the underlying number system (circular octonions) and no such consideration with respect to the time axis is needed. The Green's function can be obtained in closed form using (28) as:

$$\begin{aligned}G\left(x-y\right) &= \int \frac{d^4p}{\left(2\pi\right)^4}\frac{\sum_{\nu=0}^{3}\beta_\nu p_\nu + m}{\sum_{\nu=0}^{3} p_\nu^2 - m^2}\exp\left[-i\sum_{\nu=0}^{3} p_\nu\left(x_\nu - y_\nu\right)\right]\\ &= \left(i\sum_{\nu=0}^{3}\beta_\nu\partial_\nu + m\right)\int \frac{d^4p}{\left(2\pi\right)^4}\frac{\exp\left[-i\sum_{\nu=0}^{3} p_\nu\left(x_\nu - y_\nu\right)\right]}{\sum_{\nu=0}^{3} p_\nu^2 - m^2}\end{aligned} \tag{34}$$

The remaining integral

$$T\left(x-y\right) := \int \frac{d^4p}{\left(2\pi\right)^4}\frac{\exp\left[-i\sum_{\nu=0}^{3} p_\nu\left(x_\nu - y_\nu\right)\right]}{\sum_{\nu=0}^{3} p_\nu^2 - m^2} \tag{35}$$

is symmetric in all four dimensions of $x = (t, \vec{x})$ and $p = (E, \vec{p})$, and does not require special consideration with respect to events in the future ($x_0 < y_0$) or in the past ($x_0 > y_0$). As with the free particle solutions before, this could be interpreted as a required property of a formalism describing gravity.

5 Conclusion and Outlook

While eigenvectors and Green's function of the circular octonionic counterpart to the classical Dirac equation show invariance under time reversal, as one might expect from a quantum gravitational primitive, it remains open how gravitational interaction could be introduced into this formalism. In order to do so, basic and far reaching physical questions will need to be answered: How can one define equivalent frames of references to warrant universal applicability of physical law? How does the resulting formalism relate to General Relativity's undisputed validity for gravitation on large scales?

Use of hypernumber arithmetics could assist in answering these and other open questions. In this paper the conic sedenion relation

$$\bigtriangledown_{\text{con16}}\Psi_{\text{con16}} = 0 \tag{36}$$

was examined with focus on its circular octonionic subalgebra. Looking beyond gravitation, other hypernumber types (in particular w arithmetic) have been offered to be suitable for description of forces in physics [6]. The finding here that the simple concept of hypernumber rotation may relate to classical and new physics appears encouraging for further exploration.

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