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Dirac equation on hyperbolic octonions

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Abstract

When extending complex number algebra using nonreal square roots of +1, the resulting arithmetic has long exhibited signs for potential applicability in physics. This article provides proof to a statement by C. Musès [1] that the Dirac equation in physics can be found in conic sedenions (or 16-dimensional M-algebra). Hyperbolic octonions (or counteroctonions), a subalgebra of conic sedenions, are used to describe the Dirac equation sufficiently in a simple form. In the example of conic sedenions, a method is then outlined on how hypernumbers could potentially further aid mathematical description of physical law, by transitioning between different geometries through genuine hypernumber rotation.

Key words: counteroctonions, hyperbolic octonions, sedenions, hypernumbers, Dirac equation, countercomplex numbers, conic complex numbers

1 Introduction

The Dirac equation in physics is a fundamental quantum mechanical relation, serving as equation of motion for a free spin 1/2 particle (like e.g. an electron or neutrino). It forms an elementary building block in current description of electromagnetism, weak, and strong force. Charles Musès stated in [1] about the Dirac equation that "... a simpler version of the equation using only 16-dimensional M-algebra [1] is possible ...". This mapping of a fundamental physical relation onto a non-associative number system (in this case conic sedenions) departs from the traditional approach which uses matrix or tensor formalisms on associative (circular) complex numbers.

Detailed analysis of the pertaining hypernumber systems has been performed [2,3], from which notation and definitions are adapted here unless otherwise noted.

The Dirac equation will be written in this paper as a hyperbolic octonion product, which is a subalgebra of conic sedenions. This will provide proof to Musès' claim from above. Expressing the Dirac equation in such a non-associative arithmetic may then offer an interesting opportunity for further exploration. Genuine hypernumber rotation may qualify as a new class of symmetry transformations on the Dirac equation, and equip physicists with an additional mathematical toolset to further explore and describe fundamental relations and forces in nature. The general method will be demonstrated in the example of conic sedenions, where rotation in the $(1, i_0)$ plane allows to transition the Dirac equation from hyperbolic to circular geometry, thus "unifying" the classical relation with a hypothetical "other force".

2 Dirac Equation in Hyperbolic Octonions

In order to keep concepts from physics to a minimum in this paper, only the most common explicit form of the Dirac equation (the so-called "Dirac representation") will be examined. Physical constants c and h are set to 1 since they are non-essential for the mathematical structure.

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[1] K. Carmody recently communicated his intention to use in future writings the term *conic sedenions* instead of *sedenions* ([2,3]) or *16-dimensional M-algebra* ([1]). The new term creates a clear distinction with respect to number systems like in [4]. In a similar fashion, the term *hyperbolic octonions* will be used in this article instead of *counteroctonions* to reflect their geometric quality as compared to circular octonions (built on square roots of −1 only).

A particle's wave function Ψ is expressed as a four vector containing circular complex $\psi_\mu := \psi^{\mathrm{r}}_\mu + i\psi^{\mathrm{i}}_\mu$, with $\mu \in \{0,1,2,3\}$ and upper index $\{\mathrm{r},\mathrm{i}\}$ denoting a component's real and imaginary [2] parts. The ψ_μ are functions on space x_1, x_2, x_3 and time x_0. The abbreviation ∂_μ is short for partial derivative $\partial/\partial x_\mu$.

The classical Dirac equation can then be written as:

$$\begin{pmatrix} -m + i\partial_0 & 0 & -i\partial_3 & -i\partial_1 - \partial_2 \\ 0 & -m + i\partial_0 & -i\partial_1 + \partial_2 & i\partial_3 \\ i\partial_3 & i\partial_1 + \partial_2 & -m - i\partial_0 & 0 \\ i\partial_1 - \partial_2 & -i\partial_3 & 0 & -m - i\partial_0 \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{1}$$

A hyperbolic octonion z_{hyp8} to basis elements $b_{\mathrm{hyp8}} \in \{1, i_1, i_2, i_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7\}$ will be expressed through real number coefficients $c[b_{\mathrm{hyp8}}]$ in the following notation:

$$z_{\mathrm{hyp8}} = (c[1], c[i_1], c[i_2], c[i_3], \quad c[\varepsilon_4], c[\varepsilon_5], c[\varepsilon_6], c[\varepsilon_7]) \tag{2}$$

The particle's wave function Ψ will be mapped per definition onto:

$$\Psi_{\mathrm{hyp8}} := \left(\psi^{\mathrm{r}}_0, \psi^{\mathrm{i}}_0, \psi^{\mathrm{r}}_1, \psi^{\mathrm{i}}_1, \quad \psi^{\mathrm{r}}_2, -\psi^{\mathrm{i}}_2, -\psi^{\mathrm{r}}_3, -\psi^{\mathrm{i}}_3\right) \tag{3}$$

For any μ, if ψ^{r}_μ is mapped to $c[b_{\mathrm{hyp8}}]$ its imaginary counterpart ψ^{i}_μ is mapped to a $c[b_{\mathrm{hyp8}} \cdot i_1]$. Therefore, the circular complex imaginary basis element i is identified with the hyperbolic octonion basis element i_1, or $i \equiv i_1$.

With definition of

$$\triangledown_{\mathrm{hyp8}} := (-m, \partial_0, 0, 0, \quad 0, -\partial_3, \partial_2, -\partial_1) \tag{4}$$

the Dirac equation can be written as generic hyperbolic octonion product:

$$\triangledown_{\mathrm{hyp8}} \Psi_{\mathrm{hyp8}} = 0 \tag{5}$$

Proof: The four circular complex relations of the Dirac equation (1) are separated into their real and imaginary parts:

$$(-m + i\partial_0)\left(\psi^{\mathrm{r}}_0 + i\psi^{\mathrm{i}}_0\right) - i\partial_3\left(\psi^{\mathrm{r}}_2 + i\psi^{\mathrm{i}}_2\right) - (i\partial_1 + \partial_2)\left(\psi^{\mathrm{r}}_3 + i\psi^{\mathrm{i}}_3\right) = 0$$

$$-m\psi^{\mathrm{r}}_0 - \partial_0\psi^{\mathrm{i}}_0 + \partial_1\psi^{\mathrm{i}}_3 - \partial_2\psi^{\mathrm{r}}_3 + \partial_3\psi^{\mathrm{i}}_2 = 0 \tag{6}$$

$$-m\psi^{\mathrm{i}}_0 + \partial_0\psi^{\mathrm{r}}_0 - \partial_1\psi^{\mathrm{r}}_3 - \partial_2\psi^{\mathrm{i}}_3 - \partial_3\psi^{\mathrm{r}}_2 = 0 \tag{7}$$

$$(-m + i\partial_0)\left(\psi^{\mathrm{r}}_1 + i\psi^{\mathrm{i}}_1\right) + (-i\partial_1 + \partial_2)\left(\psi^{\mathrm{r}}_2 + i\psi^{\mathrm{i}}_2\right) + i\partial_3\left(\psi^{\mathrm{r}}_3 + i\psi^{\mathrm{i}}_3\right) = 0$$

$$-m\psi^{\mathrm{r}}_1 - \partial_0\psi^{\mathrm{i}}_1 + \partial_1\psi^{\mathrm{i}}_2 + \partial_2\psi^{\mathrm{r}}_2 - \partial_3\psi^{\mathrm{i}}_3 = 0 \tag{8}$$

$$-m\psi^{\mathrm{i}}_1 + \partial_0\psi^{\mathrm{r}}_1 - \partial_1\psi^{\mathrm{r}}_2 + \partial_2\psi^{\mathrm{i}}_2 + \partial_3\psi^{\mathrm{r}}_3 = 0 \tag{9}$$

$$i\partial_3\left(\psi^{\mathrm{r}}_0 + i\psi^{\mathrm{i}}_0\right) + (i\partial_1 + \partial_2)\left(\psi^{\mathrm{r}}_1 + i\psi^{\mathrm{i}}_1\right) - (m + i\partial_0)\left(\psi^{\mathrm{r}}_2 + i\psi^{\mathrm{i}}_2\right) = 0$$

$$-m\psi^{\mathrm{r}}_2 + \partial_0\psi^{\mathrm{i}}_2 - \partial_1\psi^{\mathrm{i}}_1 + \partial_2\psi^{\mathrm{r}}_1 - \partial_3\psi^{\mathrm{i}}_0 = 0 \tag{10}$$

$$-m\psi^{\mathrm{i}}_2 - \partial_0\psi^{\mathrm{r}}_2 + \partial_1\psi^{\mathrm{r}}_1 + \partial_2\psi^{\mathrm{i}}_1 + \partial_3\psi^{\mathrm{r}}_0 = 0 \tag{11}$$

$$(i\partial_1 - \partial_2)\left(\psi^{\mathrm{r}}_0 + i\psi^{\mathrm{i}}_0\right) - i\partial_3\left(\psi^{\mathrm{r}}_1 + i\psi^{\mathrm{i}}_1\right) - (m + i\partial_0)\left(\psi^{\mathrm{r}}_3 + i\psi^{\mathrm{i}}_3\right) = 0$$

$$-m\psi^{\mathrm{r}}_3 + \partial_0\psi^{\mathrm{i}}_3 - \partial_1\psi^{\mathrm{i}}_0 - \partial_2\psi^{\mathrm{r}}_0 + \partial_3\psi^{\mathrm{i}}_1 = 0 \tag{12}$$

$$-m\psi^{\mathrm{i}}_3 - \partial_0\psi^{\mathrm{r}}_3 + \partial_1\psi^{\mathrm{r}}_0 - \partial_2\psi^{\mathrm{i}}_0 - \partial_3\psi^{\mathrm{r}}_1 = 0 \tag{13}$$

[2] For clarity, the imaginary base element for circular complex numbers will be written as i, without index. Such indexing would make the Dirac equation hard to read and be unneeded. When using octonion and sedenion arithmetic, the circular complex i will subsequently be identified with i_1 by definition, i.e. $i \equiv i_1$.

The hyperbolic octonion product $\bigtriangledown_{\text{hyp8}}\Psi_{\text{hyp8}}$ (5) is explicitely:

$$
\begin{aligned}
&(-m, \partial_0, 0, 0, \quad 0, -\partial_3, \partial_2, -\partial_1)\left(\psi_0^{\text{r}}, \psi_0^{\text{i}}, \psi_1^{\text{r}}, \psi_1^{\text{i}}, \quad \psi_2^{\text{r}}, -\psi_2^{\text{i}}, -\psi_3^{\text{r}}, -\psi_3^{\text{i}}\right) \\
= \quad &(-m\psi_0^{\text{r}}, \ -m\psi_0^{\text{i}}, \ -m\psi_1^{\text{r}}, \ -m\psi_1^{\text{i}}, \ -m\psi_2^{\text{r}}, \quad m\psi_2^{\text{i}}, \quad m\psi_3^{\text{r}}, \quad m\psi_3^{\text{i}}) \\
&+(-\partial_0\psi_0^{\text{i}}, \quad \partial_0\psi_0^{\text{r}}, -\partial_0\psi_1^{\text{i}}, \quad \partial_0\psi_1^{\text{r}}, \quad \partial_0\psi_2^{\text{i}}, \quad \partial_0\psi_2^{\text{r}}, -\partial_0\psi_3^{\text{i}}, \quad \partial_0\psi_3^{\text{r}}) \\
&+(\quad \partial_3\psi_2^{\text{i}}, -\partial_3\psi_2^{\text{r}}, -\partial_3\psi_3^{\text{i}}, \quad \partial_3\psi_3^{\text{r}}, -\partial_3\psi_0^{\text{i}}, -\partial_3\psi_0^{\text{r}}, -\partial_3\psi_1^{\text{i}}, \quad \partial_3\psi_1^{\text{r}}) \\
&+(-\partial_2\psi_3^{\text{r}}, -\partial_2\psi_3^{\text{i}}, \quad \partial_2\psi_2^{\text{r}}, \quad \partial_2\psi_2^{\text{i}}, \quad \partial_2\psi_1^{\text{r}}, -\partial_2\psi_1^{\text{i}}, \quad \partial_2\psi_0^{\text{r}}, \quad \partial_2\psi_0^{\text{i}}) \\
&+(\quad \partial_1\psi_3^{\text{i}}, -\partial_1\psi_3^{\text{r}}, \quad \partial_1\psi_2^{\text{i}}, -\partial_1\psi_2^{\text{r}}, -\partial_1\psi_1^{\text{i}}, -\partial_1\psi_1^{\text{r}}, \quad \partial_1\psi_0^{\text{i}}, -\partial_1\psi_0^{\text{r}})
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
= (\ &-m\psi_0^{\text{r}} - \partial_0\psi_0^{\text{i}} + \partial_3\psi_2^{\text{i}} - \partial_2\psi_3^{\text{r}} + \partial_1\psi_3^{\text{i}} \ , \\
&-m\psi_0^{\text{i}} + \partial_0\psi_0^{\text{r}} - \partial_3\psi_2^{\text{r}} - \partial_2\psi_3^{\text{i}} - \partial_1\psi_3^{\text{r}} \ , \\
&-m\psi_1^{\text{r}} - \partial_0\psi_1^{\text{i}} - \partial_3\psi_3^{\text{i}} + \partial_2\psi_2^{\text{r}} + \partial_1\psi_2^{\text{i}} \ , \\
&-m\psi_1^{\text{i}} + \partial_0\psi_1^{\text{r}} + \partial_3\psi_3^{\text{r}} + \partial_2\psi_2^{\text{i}} - \partial_1\psi_2^{\text{r}} \ , \\
&-m\psi_2^{\text{r}} + \partial_0\psi_2^{\text{i}} - \partial_3\psi_0^{\text{i}} + \partial_2\psi_1^{\text{r}} - \partial_1\psi_1^{\text{i}} \ , \\
&\ \ m\psi_2^{\text{i}} + \partial_0\psi_2^{\text{r}} - \partial_3\psi_0^{\text{r}} - \partial_2\psi_1^{\text{i}} - \partial_1\psi_1^{\text{r}} \ , \\
&\ \ m\psi_3^{\text{r}} - \partial_0\psi_3^{\text{i}} - \partial_3\psi_1^{\text{i}} + \partial_2\psi_0^{\text{r}} + \partial_1\psi_0^{\text{i}} \ , \\
&\ \ m\psi_3^{\text{i}} + \partial_0\psi_3^{\text{r}} + \partial_3\psi_1^{\text{r}} + \partial_2\psi_0^{\text{i}} - \partial_1\psi_0^{\text{r}} \ )
\end{aligned} \tag{15}
$$

The eight components of (15) can be identified as left-hand part of equations (6) through (13), either identically or with the opposite sign. This proves that $\bigtriangledown_{\text{hyp8}}\Psi_{\text{hyp8}} = 0$ is indeed equivalent to the Dirac equation and therefore validates Musès' claim as stated in the introduction of this paper.

3 Hypernumber Rotation as Symmetry Transformation

In the current *Standard Model* for electromagnetism, weak, and strong interaction, physicists have successfully been able to expand the Dirac equation by adding terms that warrant invariance under certain symmetry transformations on the operand Ψ. These transformations are generally expressed in matrix form on circular complex numbers.

In a simple example, use of hypernumber arithmetic will now be suggested as additional candidate for expansion of this fundamental relation in physics. Without further speculating here about its actual relevance in describing physical law, the following is intended to be a demonstration of method only. Instead of matrix form on circular complex numbers, hypernumber arithmetic will become the genuine method of mathematical description.

A conic sedenion z_{con16} to basis elements $b_{\text{con16}} \in \{1, i_1, ..., i_7, i_0, \varepsilon_1, ..., \varepsilon_7\}$ will be expressed through real number coefficients $c[b_{\text{con16}}]$ in the following notation:

$$z_{\text{con16}} = (c[1], c[i_1], ..., c[i_7], \quad c[i_0], c[\varepsilon_1], ..., c[\varepsilon_7]) \tag{16}$$

Conic sedenions contain both a hyperbolic and a circular octonion subalgebra. The hyperbolic octonion subalgebra from above (2) will be mapped to

$$z_{\text{hyp8}} \mapsto (c[1], c[i_1], c[i_2], c[i_3], 0, 0, 0, 0, \quad 0, 0, 0, 0, c[\varepsilon_4], c[\varepsilon_5], c[\varepsilon_6], c[\varepsilon_7]) \tag{17}$$

and the circular octonion z_{cir8} subalgebra to basis elements $b_{\text{cir8}} \in \{1, i_1, ..., i_7\}$ to:

$$z_{\text{cir8}} \mapsto (c[1], c[i_1], c[i_2], c[i_3], c[i_4], c[i_5], c[i_6], c[i_7], \quad 0, 0, 0, 0, 0, 0, 0, 0) \tag{18}$$

With definition of

$$\bigtriangledown_{\text{Q1}} := (-m, \partial_0, 0, 0, 0, 0, 0, 0, \quad 0, 0, 0, 0, 0, 0, 0, 0) \tag{19}$$

$$\bigtriangledown_{\text{Q2}} := (0, 0, 0, 0, 0, \partial_3, -\partial_2, \partial_1, \quad 0, 0, 0, 0, 0, 0, 0, 0) \tag{20}$$

the operator $\bigtriangledown_{\text{hyp8}}$ from (4) and a new circular octonion counterpart $\bigtriangledown_{\text{cir8}}$ can be written as:

$$\nabla_{\mathrm{hyp8}} \mapsto \nabla_{\mathrm{Q1}} + i_0 \, \nabla_{\mathrm{Q2}} \tag{21}$$

$$\nabla_{\mathrm{cir8}} \mapsto \nabla_{\mathrm{Q1}} + \nabla_{\mathrm{Q2}} \tag{22}$$

Similarly, definition of

$$\Psi_{\mathrm{Q1}} := \left(\psi_0^{\mathrm{r}}, \psi_0^{\mathrm{i}}, \psi_1^{\mathrm{r}}, \psi_1^{\mathrm{i}}, 0,0,0,0, \quad 0,0,0,0,0,0,0,0\right) \tag{23}$$

$$\Psi_{\mathrm{Q2}} := \left(0,0,0,0, -\psi_2^{\mathrm{r}}, \psi_2^{\mathrm{i}}, \psi_3^{\mathrm{r}}, \psi_3^{\mathrm{i}}, \quad 0,0,0,0,0,0,0,0\right) \tag{24}$$

allows to map the operand Ψ_{hyp8} from (3) and a new relating Ψ_{cir8} to:

$$\Psi_{\mathrm{hyp8}} \mapsto \Psi_{\mathrm{Q1}} + i_0 \Psi_{\mathrm{Q2}} \tag{25}$$

$$\Psi_{\mathrm{cir8}} \mapsto \Psi_{\mathrm{Q1}} + \Psi_{\mathrm{Q2}} \tag{26}$$

Using a real factor α and conic sedenions ∇_{con16} and Ψ_{con16} like

$$\nabla_{\mathrm{con16}} := \nabla_{\mathrm{Q1}} + \exp\left(\alpha i_0\right) \nabla_{\mathrm{Q2}}$$

$$\Psi_{\mathrm{con16}} := \Psi_{\mathrm{Q1}} + \exp\left(\alpha i_0\right) \Psi_{\mathrm{Q2}}$$

this allows for continuous transition of the Dirac equation $\nabla_{\mathrm{hyp8}} \Psi_{\mathrm{hyp8}} = 0$ into a new circular octonionic counterpart:

$$\nabla_{\mathrm{cir8}} \Psi_{\mathrm{cir8}} = 0 \tag{27}$$

The Dirac equation corresponds to $\alpha = \pi/2$ and its new counterpart on circular geometry (27) to $\alpha = 0$.

In a physicist's perspective and wording, the conic sedenion relation

$$\nabla_{\mathrm{con16}} \Psi_{\mathrm{con16}} = 0 \tag{28}$$

would therefore unify the classical Dirac equation on hyperbolic geometry with a new relation on circular geometry by means of a mixing angle α. The expression $\exp\left(\alpha i_0\right)$ could be interpreted as symmetry transformation of a new class formed by genuine conic complex rotations.

Whether or not such hypernumber arithmetic will actually offer the benefit of describing physical law is subject to further investigation. Finding the Dirac equation in hyperbolic octonion arithmetic appears encouraging, since it could open a window to broadening traditional circular complex number arithmetic for description of the different symmetries and geometries of physical forces.

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References

[1] C. Musès, Hypernumbers and Quantum Field Theory with a Summary of Physically Applicable Hypernumber Arithmetics and their Geometries. *Appl. Math. Comput.* 6:63-94 (1980).

[2] K. Carmody, Circular and Hyperbolic Quaternions, Octonions, and Sedenions. *Appl. Math. Comput.* 28:47-72 (1988).

[3] K. Carmody, Circular and Hyperbolic Quaternions, Octonions, and Sedenions - Further Results. *Appl. Math. Comput.* 84:27-47 (1997).

[4] K. Imaeda, M. Imaeda, Sedenions: algebra and analysis. *Appl. Math. Comput.* 115:77-88 (2000)