# "1 + 1 = 2"
# A step in the wrong direction?

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(2012)

## Introduction

Fundamental questions in physics can be asked anytime, anywhere. Often they arise at the interface of physics, mathematics, and philosophy – where scrapping conversation turns into testable hypothesis. This essay explores the idea that the primitive act of counting "1, 2, 3 …" makes an implicit assumption that ultimately causes some of the challenges faced in quantum mechanics today.

A hypothesis for what could be done differently is developed during a humorous, yet serious, conversation among a physics student, a math student, an ex-philosophy student, and a city councilor. Beginning with a physics student's ill-fated attempt at bargaining for a lower price, the essay touches upon beauty in numbers and nature; repetition, inversion, and algebraic closure in mathematics; and observability in quantum mechanics.

A surprising property of the complex numbers will be shown to indicate incompleteness or inadequacy in regard to resolving certain questions in quantum mechanics. A new kind of number and arithmetic may be needed, and a proposal for such is sketched using the E8 lattice.

## Counting, repeating, and more

*Near closing time of a little store at the university, a physics student [S] selects an apple and an orange for purchase, each priced at $1. The cashier [C] at the register is about his age, and she begins the dialog.*

C: "That is two dollars."

S: *(Low on money)* "Well, that's a bit much, isn't it? One dollar for an apple is already steep, and so is $1 for an orange. Shouldn't I get a discount?"

C: “One dollar plus one dollar is two dollars. That's what it is.”

S: “Well, it's too much.”

C: “It's not too much. And now that I think about it, it is hardly enough to pay for everything involved. It should be more!”

S: “Why so? Why should it be more?” *(Little did he know what was about to unfold …)*

C: “Because the arithmetic, $1 + 1 = 2$, removes the essential difference of it being an orange and an apple. The number 2 treats them as if they were indistinguishable. Just by looking at the total price you are not able to tell how that total is made. So really, you're getting more.”

S: *(Realizing the cashier's wit and that he might be up for an intellectual tug-of-war)* “Ha-ha-ha!? Indistinguishable! Like a quantum system from two components. You're saying that there is something hidden behind $1 + 1 = 2$? Some distinguishability magic that would reveal more information upon closer look? I am a physics student, and I can prove you wrong. Quantum mechanics has no hidden variables, but your apples-and-oranges model does. Just look into the shopping basket and you know what is what.”

C: “Indeed, we have a way of observing the shopping basket and of understanding the true nature behind the number 2 in that respect.”

S: “So one dollar each, making two dollars total. It's too much.”

C: “How could you be so simple-minded! Imagine you would come here every day, 7 times a week, 12 months a year, do you understand what a wealth of possible purchases you could make? Even if you would only select between an apple and an orange, the possible combinations are amazing …”

S: “You mean by multiplying all the different possibilities?”

C: “Yes! When you add apples to your basket you are repeating a seemingly simple process: $1 + 1 = 2$, then $2 + 1 = 3$, and so on. But that is only how it looks on the surface – when you only care about the special case where your objects are indistinguishable. Once you accept that any two objects differ by their very nature, this topic becomes amazingly rich and faceted! You can't just mix apples and oranges.”

C: *(Continuing …)* “So, if you came tomorrow to do another purchase, and the day after that, and so on – what you called ‘multiplication’ – is essentially the same thing as addition: It's repetition of indistinguishable events, or entities.”

S: *(Chiming in)* “And there's exponentiation …”

C: “Same thing! Just keep inviting more people to do it with you. So 5 to the power of 3 could describe 5 fruits purchased by 5 students, 5 times. And 5 to the power of 4 would be that repeated in 5 different cities … Just keep on going, there's no end to it.”

S: *(After a short thinking pause)* “Still, that's a lot of money.”

*(C just shakes her head)*

S: “Oh, I get it! Just as we repeat addition to get multiplication, and repeat multiplication to get exponentiation, we can continue this on and on. I've heard of that – it's called making ‘power towers’. You get really huge numbers with that – Graham's number, Knuth's arrow notation …”

C: “Stop, stop!”

S: “Why? Why can't I just go on like that?”

C: *(With a calm voice)* “It doesn't keep going like that if you understand numbers in the reality they create.”

S: *(Puzzled; he now begins to doubt that rational debate may be possible here …)*

C: *(Unaffected by S' surprised look)* “Not only have you completely forgotten the essence of apples and oranges, you're now treating numbers like a one way street: ‘Just keep going on and on, and repeat and repeat. Surely this must go on forever.’ So thinks the child. How naïve. Instead, inverse functions are essential to nature, and you don't have good closure spaces if you just keep going.”

*(S is getting irritated with C)*

C: *(Lifts her nose and speaks sternly with a somewhat blank stare as if there was something far away)* “The nature I observe is the reality I live and think in. It is a nature that reflects into itself. Whatever you add can be removed. These are necessary dual operations.”

S: *(Barking back)* “Well, I think you are naïve. You claim: ‘Whatever you remove can be added.’ By that reasoning, you could argue: If there are 10 people on a bus, and 15 leave, then 5 people will have to get back on so that the bus is empty. That philosophy doesn't last long.”

C: *(Getting introverted)* “There is a greater truth in numbers than your little pun makes fun of. In fact, numbers are truth, and arithmetic is the universe.”

S: *(Baffled. The conversation has become personal)* “I think you've lost your marbles.”

C: “No I haven't! I'm right. Why else would I study pure mathematics?”

S: “You … what?! … That doesn't follow … Your area of study is your own choice – not the other way around. This is crazy!”

C: “I'm afraid I'm going to have to call the Manager.”

## Complex numbers do it all?

*The store manager [M] enters with a concerned but caring look.*

M: “What's going on?”

C: “This physics student believes that $2 is too high a price to pay for a $1 apple and a $1 orange. He doesn't understand how much of a bargain he gets by reducing the essence of apple and orange to a simple number, 2.”

M: *(Sighs)*

C: “We spoke on this topic last week, remember?”

M: *(Slowly)* “I remember last week, and the week before, and the week before that.” *(He addresses S)* “Young man, I'm afraid you're facing a battle that you can only loose.”

S: “We'll see about that. I've just passed my finals in quantum mechanics with a straight A, and this young lady here tells me about hidden variables mysticism in apples and oranges. And even more, we all know that repeated addition leads to multiplication, repeated multiplication to exponentiation, repeated exponentiation to power towers, and so on. Everyone knows that! But she disagrees. Why? I don't know.”

M: *(Even slower)* “You may be right about repetition, but did you think about inversion of an operation? Did you think about the closure space?"

S: *(His jaw drops. Clearly, he thinks, M and C are teaming up against him)*

C: “For every addition there is a subtraction. For every multiplication there is a division.”

S: *(In cheeky hope)* “But not for division by zero!”

C: *(Overcome with the blank stare again)* “Yes, Zero … Someday we'll understand why she is special: as a point, as nothing, as everything …”

M: *(Interjects)* “Forget zero; zero purchase, zero money – that means I've got zero interest in this all. If there's nothing, then you can't repeat it.”

C: “You're talking again as if Zero is nothing. But what if Zero is really everything?”

M: *(Getting restless)* “Well if zero's everything then you can't repeat that either, can you?”

*(S grins broadly, while C has the sad look of defeat from facing an overwhelming majority, if wrong)*

S: “So let's go down your line of thought, and say that nature has a knack for repetition, inversion, and closure. Why not? I kind of like that myself! Still, it's as simple as $1, 2, 3$ that becomes $-1, -2, -3$ under an additive inversion. This requires supplying opposites to the naturals, the negatives. Alas, the integers are closed under addition and subtraction. That's no magic; that's simply $1 + 1 = 2$.”

C: *(Bored)* “Keep on.”

S: “Well, I'm sure I'm right! You formalize repeated addition of the same number and call it multiplication. No big deal, still.”

C: *(Mumbling to herself)* “It's no big deal if you ignore the broken symmetry: $(+1) * (+1)$ is $+1$, but $(-1) * (-1)$ is also $+1$, as if $+1$ would be something special.”

S: “No big deal. You invert multiplication to get division, and make it complete or … what did you call it again?”

M: *(Helping out)* “The closure space.”

S: “Right, you make division closed by creating the fractions, and you call that result ‘the rational numbers’. They're the closure space for the division operation on the integers.”

C: “You forget Zero.”

S: “Well, it works most of the time. Besides, $0$ is just a point.”

C: “Zero – just a point!” *(She is almost yelling)*

S: *(Backing off a bit)* “I mean $0$ is a special case. I'm sure we can deal with that. Moving on to exponentiation, roots, and logarithms, we have the complex numbers. Done. So what's next?”

M: *(Putting on a comforting but concerned look)* “Please, what a great line of thought to mull over, by ourselves at home.”

C: *(Suddenly furious)* “You ask ‘What's next’? What's next is your biggest illusion! It’s an illusion about exponential closure which you so childishly accepted from your studies. I bet those good grades only confirmed your own deranged view of mathematics! Does knowing how to solve some quantum equation make you feel powerful? Ah – you think you know it all!”

M: “Please …”

C: *(Reasserting herself strongly)* “Well, you don't know it all. You use complex numbers in your physics, right? Repeated multiplication leads to exponentiation, and so on. And you really think that complex numbers are the necessary closure space for exponentiation?”

S: *(With a smirk)* “The complex numbers are the algebraically closed field for all there is, are they not?”

C: “You've just picked up that phrase in math class: ‘algebraically closed field’. Right? As a typical physicist, you magically extended the scope of a completeness concept to everything, to ‘all there is’!” *(Then, in a momentary display of compassion)* “But, to be fair, I do understand that by ‘closed’ you mean that the complexes are algebraically complete. They are complete in that any polynomial you can create with them has all its roots within the complexes. You can't create a polynomial problem which cannot be solved within itself. That makes the complexes quite predictable, limited, and boring.”

S: *(His smirk is fading a bit)*

C: “Just because you have algebraic completeness doesn't mean that the complexes are closed or complete ‘for everything’.”

S: *(Focusing on the math he knows)* “Every polynomial equation has complex roots, and every logarithm of a complex has a solution, except for $0$. So it's all closed, or complete, or whatever you mathematicians call it.”

C: “Well, I grant you that the square root of $1$ has a solution in the complexes. Actually, it has two possible solutions: $-1$ and $+1$. And, what about the fourth root of $1$, or $1$ to the power of $(1/4)$? Shouldn't this be associated with a solution set that is made from the four values: $1, -1, i, -i$ ? Each of these to its fourth power is $1$.”

S: “That's just because you don't have proper conventions.”

M: *(Again trying to end the conversation)* “Ah yes, conventions are great. How about we re-convene tomorrow? A bit of time surely will help.”

C: *(ignores M)* “You have been brainwashed very efficiently, Mr. physics student!”

S: *(Confident what he has learned)* “Ah, I know what you mean. Logarithms can have an infinite number of distinct solutions in the complexes. For example, the logarithm of $1$ is $0$ by convention, because the Euler number $e$ to the power of $0$ is $1$: $e^0 = \exp(\ 0\ ) = 1$.”

C: “So,

$$\exp(\ 0\ ) = 1,$$

and its inverse,

$$\ln(\ 1\ ) = 0.$$

Now take $2\pi N$ multiples of the imaginary $i$, $2\pi iN$, where $N$ is an integer. Insert any of those multiples as the exponent, instead of $0$, and you also get:

$$\exp(\ 2\pi iN\ ) = 1.$$

So we could also have

$$\ln(\ 1\ ) = 2\pi iN,$$

by definition of inverse. What does this mean?”

S: “The result of a function could be more than one number?”

C: “And why is that not a problem for you?”

S: “Why? Of course it's not a problem because we have a preferred solution. We prefer the one that is real, $\ln(\ 1\ ) = 0$, because it is simplest. You said it yourself earlier: Simplicity is key. Right? Gotcha!”

C: “Wrong.”

S: “I bet you $10 that you're the one who is wrong, not me.”

C: “Wager accepted.”

S: “Then prove it! I'm listening.”

M: *(Giving a notable sigh of relief. Ten dollars is not so bad, he thinks, it could have come to much worse)*

C: “The logarithm is the inverse function of the exponential, right?”

S: “Right.”

C: “So if I take any number $x$ and take the exponential of the logarithm of $x$ then we'll arrive at the same number: $\exp(\ \ln(\ x\ )\ ) = x$.”

S: “Sure.”

C: “If the complex numbers are, in fact, so complete for everything as you claim, then how about choosing $x$ simply to be the imaginary unit $i$. Let $x = i$.”

S: *(Thinks on this a bit)*

C: “So we say that $\exp(\ \ln(\ i\ )\ )$ is also identical to $i$. Then we can look at the simple expression of $i$ to the power of itself, $i^i$.”

S: “Woah! That is twisted. An imaginary number to the power of another imaginary number – that's totally bizarre!”

C: “If you claim completeness ‘for all there is’, as you do, then that's a perfectly normal expression; as $i$ is just another complex number. Let's replace the $i$ in the base of $i^i$, with $\exp(\ \ln(\ i\ )\ )$. That yields:

$$i^i = \exp(\ \ln(\ i\ )\ )^i.$$

Then I hope you agree that we have:

$$\exp(\ \ln(\ i\ )\ )^i = \exp(\ \ln(\ i\ ) * i\ ).”$$

S: “Well, since

$$\exp(\ \ln(\ 2\ )\ )^2 = \exp(\ \ln(\ 2\ ) * 2\ )$$

in the real numbers, this rule should hold here, too.”

C: *(With the happy smile of victory)* “Then, tell me what is the preferred expression of:

$$i^i = \exp(\ \ln(\ i\ ) * i\ )?$$

The $\ln(\ i\ )$ can be expressed as any

$$\ln(\ i\ ) = \pi i(\ 1/2 + 2N\ ).$$

By substituting, we obtain the identities:

$$i^i = \exp(\ -\pi\ (\ 1/2 + 2N\ )\ )$$
$$= (\ e^{-\pi}\ )^{(\ 1/2\ +\ 2N\ )}$$

for any integer $N$. Since $e^{-\pi}$ is an irrational real, each such solution in $N$ is also an irrational real! Tell me, which one is the *preferred* solution here? Which is the simplest ‘by convention’?”

S: “Wow, we went from one point to an infinity of points!”

C: “Yes! Who likes a system where solutions proliferate like that?!”

S: *(Amazed; he could not give a 'preferred' solution, nor was he carrying enough money to pay up for a lost bet)*

M: *(Swiftly)* “OK, you two, we have to close the store.”

## Quantum theory not so natural

*M's hope of closing up evaporates a second later when a well-dressed lady [L] rushes in.*

L: “Thank goodness you're still open.” *(She is a bit out of breath)* “Do you have some water? My throat is itching from all the talks.”

M: *(Recognizing her immediately, reaches her a bottle)* “Hello, Councilor! What gives us the honor of your visit? Running for another term on City Council?”

L: “Thank you for the water and your warm greeting! Yes, we were swinging by the university today.” *(Taking a few sips from her drink*) “So you were talking science, I heard?”

M: “Numbers, really – just numbers.”

C: *(Mocking M's downplaying tone …*) “Just complex numbers.”

L: “Oh, I dreaded those in class.”

S: *(Cutting to the chase as if this is what people talk about at City Council)* “Quantum mechanics is unthinkable without complex numbers. In fact, they are so intimately tied to quantum mechanics that one could even …” *(addressing C and raising his voice a bit)* “… think that Math is Nature.”

L: “How wonderful, wonderful! Did you know that it was my support that landed the federal grant for our ‘Center of Quantum Information’? Isn't it amazing what technology can do for us if only we sponsor the right people?”

C: *(In a matter of fact tone, politically insensitive)* “These complex numbers are a flawed math for the task, therefore your ‘Quantum Center’ is at an end before it even starts.”

L: “I beg your pardon?”

S: *(Interjecting)* “Councilor, this young lady is a bright and insightful math student. She believes that complex numbers are not quite the right tool to describe nature.”

L: “May I point out that we do put a lot of thought into funding significant research. Expert opinion is essential in helping us understand what fields of research are promising for the future of this great city.” *(Pausing briefly)* “What exactly is your concern, as a math student?”

C: “Well, we use numbers for indistinguishable things. For example, $1 + 1 = 2$ is saying: ‘Here's one thing, there's one thing, and together that makes two things.’ That may work well for many aspects of nature, but in reality we can't just claim everything is indistinguishable.”

L: “Aha? Please go on.”

C: “You see, we invent addition as repetition of the same $1$ thing. And we invent its inverse operation, subtraction. Then we supply positive and negative numbers to provide closure within the set.”

L: “Sure, we need these numbers to use them all fully.”

C: “Yes. Then we invent multiplication and its inverse operation, division. We supply the fractions for closure. We go on to invent exponentiation and its inverse operations: logarithm and root-taking. We want closure for things like the square root of $-1$, so we supply complex numbers for that.”

L: *(Surprisingly adept at listening)* “OK, I think I follow your thought. It's about some analogy between different operations in math.”

C: “That's it, exactly! And all along we required inverse functions and supplied new numbers to obtain closure, or a maximal set for widest use. Using your words, ‘we need these new numbers to use them all fully.’ Quite fitting.”

L: “Good. So far this all seems fine, but I overheard your rather unique comment earlier, as I entered the store. What was that again?"

S: *(Seizes his chance to fight for his $10 that are still at stake from the earlier bet)* “Yes, my friend did bring up an interesting example of what it would mean to take the imaginary unit $i$ to the power of $i$. And I must admit that, until now, I hadn't thought about this being somewhat of a … problem.”

L: “Why would this be a problem for our ‘Center of Quantum Information’? That's really what I'm driving at.”

S: “Well, since quantum mechanics is so closely tied to complex numbers, it is a bit confusing to me to learn that the math isn't quite as ‘complete’ as I thought.”

M: *(Although he hasn't contributed much to the discussion, he suddenly seems interested and engaged)* “If I may say, Councilor, this should be one of the interesting things for the Center to find out. You see, when I studied philosophy, the foundations of quantum mechanics always interested me.” *(S is surprised and amazed at M’s revelation.)* “But academia was not quite my place, so I took over this store from our family business – and, happily, enjoy my independence. But there is something about quantum mechanics that has never settled well with me: the randomness aspect in the observation.”

L: “Well, it's not all random, is it?”

M: “Maybe. Quantum mechanics describes quantum systems through wave functions that are unobservable in principle.”

S: “And these wave functions use complex numbers like fish use water.”

M: “Precisely. In order to do any kind of observation, we must prepare a probability distribution. We say that the result is random, but weighted by certain probabilities.”

L: “Sounds familiar. We can't predict things exactly, but only within well-defined bounds. ‘Heisenberg uncertainty’, right? If you'll excuse my name dropping.”

S: “Exactly!”

M: “And it is that randomness that has never settled well with me. Why would nature be that way? It doesn't seem right.”

L: “If I may be so direct and point out that nature is the way it is without really caring what we think about it.”

C: *(A bit lightheaded)* “Imagine a magician asking the audience to pick measurement values at random, weighting them by a distribution function, performing a large number of measurements, and – alas – the magician gets a lot of ‘Ooohs’ and ‘Aaahs’ as the prediction unfolds. That's magic, not science!”

M: “You see, Councilor, we seem to have no problem saying that wave functions are unobservable, and that measurement results just so happen by random.”

L: “But that's what we observe, right?”

S: “So why would complex numbers be the problem? Why would there be a problem at all?”

*(All eyes now turn to C. The stage has been set, there is a notable tension in the small store …)*

C: *(Speaking slowly)* “Wave functions in quantum mechanics are unobservable. When observing a quantum system you must do one of those crazy things that physicists do: ‘destroy the wave function’, or interpret it all as ‘split into many worlds’. You must break your system. My hypothesis is that these procedures are artifacts that stem from using numbers $1, 2, 3$ ... that count indistinguishable things. You don't know which $1$ is the one object and which $1$ is the other. The natural numbers have unobservability built into them. Consequently, when you use those kinds of numbers in quantum mechanics to describe nature, you end up with a model that reflects their inherent unobservability."

S: *(Shrugs)* “Numbers are a tool for the model, not the other way around. Because a model behaves oddly in certain cases does not reflect back on the numbers used to describe the model.”

C: “Math is Nature, and Nature is Math. By using complex numbers for quantum mechanics, we can't get any deeper insight.”

M: *(Adds, with an encouraging smile)* “That is your hypothesis. It makes ‘$1+1=2$’ a step in the wrong direction.”

L: “So what do you propose should be changed? What should be looked at that isn't being investigated today?”

## Getting to the point, making a lattice

*There was a moment of silence in the store as everyone realized that the discussion had reached a turning point. It is easy to point out deficiencies and complain, but it is a world harder to come up with a proposal on how to fix it all.*

C: *(Looking at S and M)* “Do you remember how we started? We were counting 1, 2, 3.”

S: “Well yes.”

C: “We repeated to add, we inverted to subtract, we provided a closure space to make use of it all. Repeat, invert, close – these are our guiding principles.”

M: “That is *your* hypothesis. Very nicely stated.”

C: “Nature is both simple and beautiful.” *(M and L exchange a quick grin)* “We repeat addition to multiply, and repeat multiplication to exponentiate. We take logarithms and roots to invert such exponentiation. Complex numbers are a closure space to do a lot of arithmetic this way.”

S: “And we can do quantum mechanics with it.”

C: *(Quickly)* “All of which is flawed!”

M: *(Asserting professionalism)* “Therefore, you propose …”

C: “I propose that we look at the very beginning of number development, at the choice of separating numbers like $1,\ 2,\ 3$ … from the arithmetic we impose on them.”

L: “And just how do you want to do this?”

C: “Numbers and arithmetic need to become interchangeable duals, or holographic projections into themselves.”

L: “For the layperson, please?”

C: “Let me try. Rather than looking at numbers as isolated points upon which some external arithmetic acts, we need to give each number an internal structure as rich as the arithmetic for which you use them.”

L: “You want to use something bigger and more complicated than ordinary numbers? And then use that for new mathematics? I don't quite see how that fits into your claim that ‘Nature is both simple and beautiful’. To the contrary.”

C: “Well, Madam Councilor, the trick is to start at the very basics with developing a system using ‘repeat, invert, close’ as guiding principles. A number needs to remain a number, and arithmetic needs to remain arithmetic. Both, number and arithmetic, have to reflect into one another. Of course, the implementation is so simple: We use lattices!”

S: “You mean crystal-like lattices? I know those, you can describe such lattices through their lattice constants, where each lattice constant tells you at what distance apart the coordinate points repeat.”

C: “Yes, for simplicity, we can stay with crystalline lattices.”

S: “And you can build reciprocal lattices from those lattice constants, which model the lattice's periodicity. The lattice constant defines the lattice’s distances, like between coordinate points.”

C: “Like, for sure.” *(Mocking the student)* “Mathematicians call this concept the ‘dual lattice’. There are types of lattices that are self-dual, where the dual lattice is the same as the original lattice. In my proposal we build numbers and arithmetic on such self-dual lattices. This way we don't just build something bigger and more complicated that needs to be hidden inside some mystery container called ‘number’. Instead, we build something concise, self-reflexive, beautiful and simple.”

L: “I'm not sure that I would use the term ‘simple’ anymore, but I see how you're making very few assumptions. How can a lattice be used to do math?”

S: “And what does this have to do with $i^i$ from earlier?”

C: “Earlier we saw how a point $i$ , to its $i$-th power, leads to an infinite point-set: $\{\ i^i\ \}$. Rather than defining such a solution set away and hiding it as a special case, we embrace it: Like crystalline lattices, our new numbers are made from infinitely many discrete points.”

L: “Hmm. So, how *do* you add?”

C: “Well, addition is simple: Using a crystalline lattice as a generalized point or number within our space, such a point-lattice consists of a set of coordinate points. If we take any single coordinate point in the ambient space and add it to every other coordinate point of the lattice, we get essentially the same lattice.”

S: “Isn't the resulting lattice sum shifted relative to the original lattice?”

C: “Exactly! The point-lattice can be subtly shifted, or translated. But it remains the same lattice in that shifting is a simple transformation that maps the lattice back into itself. In that respect, we now define such shift to be ‘addition’ in our arithmetic, and rotation to be ‘multiplication’.”

L: “So I can do any kind of addition and multiplication?”

C: “You're close. In order to keep the system concise, we need to make one more restriction. When taking a lattice and building the set of points from multiplying any two points with one another, this has to be an automorphism as well.” *(She quickly clarifies…)* “It has to be a multiplication that maps the lattice into itself. We don't want the set of points to proliferate with each pairwise multiplication.”

S: “So what. I bet there are lots of lattices that do that.”

C: *(Putting on her biggest grin)* “Nope! There aren't. In fact, there are only very few lattices that allow you to multiply like that.”

*(L is not equipped to handle any more detail. She starts looking at her calendar)*

C: “The interesting lattices exist only in 2, 4, and 8 dimensions. All others don't work.”

S: “Aha?”

C: “Really. You can build a lattice in two dimensions from complex number multiplication. Just take the lattice of all integral coordinates in the 2D plane and multiply them as if they were complex numbers. In 4D you can build a similar lattice from integral quaternions, and in 8D you use integral octonions.”

L: “What are quaternions and octonions? I'm afraid I have to leave really soon so please make it short.”

C: “Quaternions and octonions are number systems in 4 and 8 dimensions, respectively. These systems are division algebras, meaning they allow you to multiply and divide by any nonzero number.”

L: “So, to wrap this up, you are proposing to use certain kinds of lattices as a new kind of number, and then build some math on it. Did I get this right?”

M: “Yes, you did – very well summarized, Councilor!” *(Clearly, he wants to go home as it has become quite late)*

L: “And you have some lattices that let you add and multiply and divide? Forgive me if I'm overly simplifying here, but is there some key word or phrase that I can use, to pass this concept on to other scientists I may meet?”

C: “Simply say: ‘E8’. The lattice in eight dimensions that works the way we need is called the ‘E8 lattice’. The other two aren't interesting.”

M: *(Quickly clarifies)* “The other two are 2D and 4D lattices. Since we are interested in the widest possible lattice, we're looking at the E8 first to build our new arithmetic on. Some of the symmetries of the E8 lattice, in turn, are the ones we observe in nuclear decay and high-energy experiments. So the E8 lattice could be big enough for modeling fundamental physics.”

L: “Well, I can remember E8 – that's where the black king starts in chess. Thank you all so much for your patience with me, it has been most inspiring. Have a good day, and remember to go and vote!”

## To zero, and beyond

*After L leaves, the store becomes very silent.*

S: “Please explain to me again how your new numbers will help in quantum mechanics?”

C: “You don't start off with indistinguishable numbers $1 + 1$ that model some unobservable entities. Instead, each lattice number is a projection of your entire configuration space into itself, like a hologram. Number and arithmetic interact with one another, just as components in a quantum system do. You're starting off right.”

S: *(Getting impatient)* “So how do I actually do quantum mechanics with your holographic numbers?”

C: *(Snaps back)* “I'll tell you when I'm done with my work!”

S: *(Seizing his chance)* "Great! Just let me know, then you'll get your $10 from our bet."

*He puts $2 onto the counter, takes apple and orange, and rushes out.*

*M gives a brief nod to C, smiles, and slowly walks into his store office. C is very satisfied. To her, the silence in the now empty store is the wonderful sound of discovery yet to be made.*

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