

# AN ANALYSIS OF RICK LOCKYER'S "OCTONION VARIANCE SIEVE"

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ABSTRACT. In "Octonion Algebra and its Connection to Physics" [14] an algorithm on octonions is brought forward for description of physical law, the "octonion variance sieve process". This paper expresses the used algorithm in symbolic form, and highlights the structure between the "function", "distance", and "algebraic invariant" concepts therein.

## 1. INTRODUCTION

Maxwell electromagnetism has been expressed various times on octonionic algebras (e.g. [1, 2, 3, 4]), and octonionic Dirac equations or spinors (e.g. [5, 6, 7, 8, 9, 10, 11, 12, 13]) promise usefulness of octonions across all fundamental forces in physics. A recent proposal, the "octonion variance sieve" in [14] introduces a class of functions on the octonions that are invariant under a new algorithm. A "law of octonion algebraic invariance" is given, that recovers the general electromagnetic action when assigning the electromagnetic field to certain octonion functions and modeling dynamic interaction through octonionic differential operators.

The current paper restates this octonion variance sieve in a more symbolic form, and highlights structural similarities between concepts used therein. Generalization to operator algebra or connection to physics (as then treated in [14]) will not be handled here.

## 2. OCTONIONS

The octonions  $\mathbb{O}$  are the highest dimensional normed division algebra. They supply  $\mathbb{R}^8$  with a multiplicative norm  $\|\cdot\| : \mathbb{R}^8 \rightarrow \mathbb{R}_0^+$ , where for any  $a, b \in \mathbb{O}$  there is  $\|a\| \|b\| = \|ab\|$ . The multiplicative inverse for any number other than 0 is unique,  $\forall a \in \mathbb{O} / \{0\} \exists b, ab = 1$ . Using eight orthogonal vectors in  $\mathbb{R}^8$  as octonion basis,

$$(2.1) \quad b_{\mathbb{O}} := \{1, i_1, \dots, i_7\},$$

an octonion is described through real coefficients  $a := (a_0, \dots, a_7)$ . "Addition" is the vector space addition, "multiplication" distributes over addition and is described by basis element relations  $i^1 = 1, i_n^2 = -1$  ( $n = 1, \dots, 7$ ), and a set of seven associative anticommutative ordered triplets  $t_{\mathbb{O}[N]}$ :

$$(2.2) \quad i_l i_m = \epsilon_{lmn} i_n \quad \text{for all } \{l, m, n\} \in t_{\mathbb{O}[N]}.$$

These associative triplets then fully describe an octonion algebra, e.g.:

$$(2.3) \quad t_{\mathbb{O}[0]} := \{\{1, 2, 3\}, \{7, 6, 1\}, \{5, 7, 2\}, \{6, 5, 3\}, \{1, 4, 5\}, \{2, 4, 6\}, \{3, 4, 7\}\}.$$

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Even permutations within a triplet are equivalent. A total of 16 sets of triplets,  $t_{\mathbb{O}[N]}$ ,  $N = 0, \dots, 15$  will now be used to describe 16 corresponding octonion multiplication rules  $\mathbb{O}[N]$  that are achieved through odd permutation within some of the triplets. Notation will be abbreviated for the  $t_{\mathbb{O}[N]}$  by using the octonion index numbers from  $t_{\mathbb{O}[0]}$  (equation 2.3) as a reference, and then indicating whether permutations within each triplet are of even (+) or odd (-) parity. For example:

$$(2.4) \quad t_{\mathbb{O}[1]} := \{+, +, -, -, +, -, -\} \\ \equiv \{\{1, 2, 3\}, \{7, 6, 1\}, \{5, 2, 7\}, \{6, 3, 5\}, \{1, 4, 5\}, \{2, 6, 4\}, \{3, 7, 4\}\}.$$

### 3. AUTOMORPHISMS

Four duality automorphisms,  $\mathcal{T}_0, \dots, \mathcal{T}_3$ , act on the octonion multiplication rules  $\mathbb{O}[N]$ :

$$(3.1) \quad \mathcal{T}_n : \{\mathbb{O}[N]\} \rightarrow \{\mathbb{O}[N]\}, \quad n \in \{0, 1, 2, 3\},$$

$$(3.2) \quad \mathcal{T}_n \mathcal{T}_n = (\text{id}),$$

$$(3.3) \quad \{(\text{id}), \mathcal{T}_n\} \cong \mathbb{Z}_2.$$

$\mathbb{Z}_2$  is the cyclic group with two elements. Acting on the  $t_{\mathbb{O}[N]}$ , the  $\mathcal{T}_n$  either leave the parity of a permutation triplet unchanged, (id), or swap it, (sw):

$$(3.4) \quad \mathcal{T}_0 := \{(\text{id}), (\text{id}), (\text{id}), (\text{id}), (\text{sw}), (\text{sw}), (\text{sw})\},$$

$$(3.5) \quad \mathcal{T}_1 := \{(\text{sw}), (\text{sw}), (\text{sw}), (\text{sw}), (\text{id}), (\text{id}), (\text{id})\},$$

$$(3.6) \quad \mathcal{T}_2 := \{(\text{id}), (\text{sw}), (\text{id}), (\text{sw}), (\text{sw}), (\text{id}), (\text{sw})\},$$

$$(3.7) \quad \mathcal{T}_3 := \{(\text{id}), (\text{id}), (\text{sw}), (\text{sw}), (\text{id}), (\text{sw}), (\text{sw})\}.$$

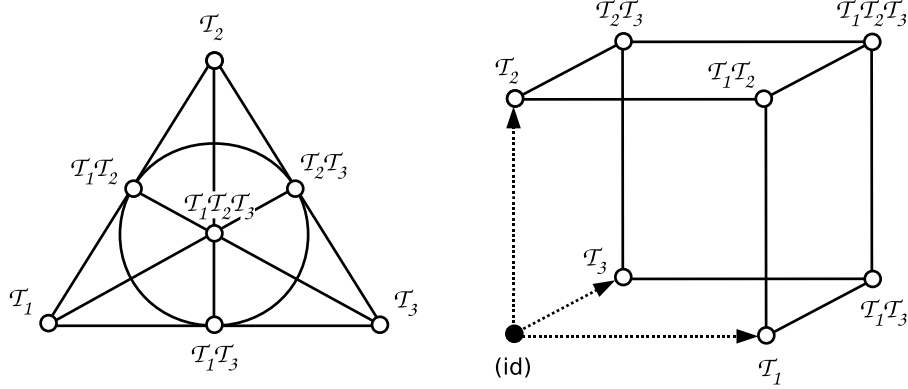
Whereas  $\mathcal{T}_0$  changes the parity of three triplets, the  $\{\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3\}$  each change the parity of four triplets.  $\mathcal{T}_0$  transitions between what is called "left-" and "right-handed" octonion multiplication rules [14], that are "not isomorphic" in the sense that they cannot be transformed into one another through transformation of the basis vectors in  $\mathbb{R}^8$  alone. Instead,  $\mathcal{T}_0$  is an algebra isomorphism that corresponds to complex conjugation, i.e., inverting the sign of all seven nonreal octonion elements. Since it changes a left-handed octonion multiplication rule into a right-handed rule, this can be called a change in "chirality", or a mapping between "opposite algebras" [5].

For a select  $n$ , the pair  $\{(\text{id}), \mathcal{T}_n\}$  forms the two element cyclic group  $\mathbb{Z}_2$  under repeat application. The possible unique combinations of the  $\{\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3\}$  form the set

$$(3.8) \quad \{\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_1\mathcal{T}_2, \mathcal{T}_1\mathcal{T}_3, \mathcal{T}_2\mathcal{T}_3, \mathcal{T}_1\mathcal{T}_2\mathcal{T}_3\}$$

which transitions between octonions  $\mathbb{O}[N]$  of the same chirality. It can be graphed in the Fano plane, where three automorphisms lay on each line such that the combination of any two automorphisms yields the third one (figure 3.1). Together with the identity element, (id), this forms the group  $\mathbb{Z}_2^3 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ .

FIGURE 3.1. All unique automorphisms from repeat application of the  $\{\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3\}$  can be graphed in the Fano plane (left), where the product of each two automorphisms on a line yields the third. Together with identity (id) this forms the group  $\mathbb{Z}_2^3 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$  (right).



All possible combinations of the  $\{\mathcal{T}_n\}$  acting on  $t_{\mathbb{O}[0]}$  then generate the 16 triplet sets  $t_{\mathbb{O}[N]}$  for the  $\mathbb{O}[N]$  respectively. Written out, the first eight are:

$$\begin{aligned}
 (3.9) \quad t_{\mathbb{O}[1]} &:= \{+, +, -, -, +, -, -\} = \mathcal{T}_3 t_{\mathbb{O}[0]}, \\
 t_{\mathbb{O}[2]} &:= \{+, -, +, -, -, +, -\} = \mathcal{T}_2 t_{\mathbb{O}[0]}, \\
 t_{\mathbb{O}[3]} &:= \{+, -, -, +, -, -, +\} = \mathcal{T}_2 \mathcal{T}_3 t_{\mathbb{O}[0]}, \\
 t_{\mathbb{O}[4]} &:= \{-, -, -, -, +, +, +\} = \mathcal{T}_1 t_{\mathbb{O}[0]}, \\
 t_{\mathbb{O}[5]} &:= \{-, -, +, +, +, -, -\} = \mathcal{T}_1 \mathcal{T}_3 t_{\mathbb{O}[0]}, \\
 t_{\mathbb{O}[6]} &:= \{-, +, -, +, -, +, -\} = \mathcal{T}_1 \mathcal{T}_2 t_{\mathbb{O}[0]}, \\
 t_{\mathbb{O}[7]} &:= \{-, +, +, -, -, -, +\} = \mathcal{T}_1 \mathcal{T}_2 \mathcal{T}_3 t_{\mathbb{O}[0]}.
 \end{aligned}$$

These correspond to the eight left-handed multiplication tables from [14]. The right-handed ones are then obtained from:

$$(3.10) \quad t_{\mathbb{O}[N+8]} := \mathcal{T}_0 t_{\mathbb{O}[N]}.$$

Overall, this concept is identical to the group action  $T$  from [5] (equation 30 therein). Octonions that are here mapped through  $\mathcal{T}_0$  are called "opposite algebra" in [5] (equation 33 therein), and correspond to octonionic spinors of opposite chirality. The structure of octonion algebra and its relation to  $\mathbb{Z}_2^3$  and Hadamard transforms is also investigated in [15].

#### 4. FUNCTIONS, DISTANCES, AND "OCTONION VARIANCE SIEVE"

This section now defines a set of 16 polynomial functions,  $A(f) := \{f[N]\}$ , that use octonion multiplication rules  $\mathbb{O}[N]$  on a given polynomial  $f$ . A certain linear

superposition of these functions, the Hadamard transform, will yield 16 "distances"  $B(f)$  (corresponding to the 14 "distances" and two "invariants" from [14]). Applying the same superposition rule again on the distances yields the original functions, making them dual to each other. Furthermore, the automorphisms on the distances  $B(f)$  are similar to the automorphisms on the octonion rules  $\mathbb{O}[N]$  used to generate the functions  $A(f)$ . On a sidenote, summing over results obtained from different multiplication rules is also part of the two dimensional "W space" [16].

Let  $f \in P$  be polynomial with finite number of arguments,  $A$  the spin functor that turns the polynomial into a polynomial function in  $\mathbb{R}^8$ , then  $f[N]$  are the functions that use the multiplication rule  $\mathbb{O}[N]$  for multiplication:

$$(4.1) \quad \{f[N]\} := \{\mathbb{R}^8 \otimes \dots \otimes \mathbb{R}^8 \rightarrow \mathbb{R}^8, f[N] \in \mathbb{O}[N]\},$$

$$(4.2) \quad A : P \rightarrow \{\mathbb{R}^8 \otimes \dots \otimes \mathbb{R}^8 \rightarrow \mathbb{R}^8\}.$$

An  $f[N]$  can then be represented by the resultant vector made from general coefficients of the function's parameters. Because octonions are a normed division algebra, with unique multiplicative inverses and free from zero-divisors or nilpotents (except for 0 itself), knowledge of all coefficients from a general octonion product allows to uniquely identify the multiplication rule  $t_{\mathbb{O}[N]}$  used<sup>1</sup>.

Given a polynomial  $f$ , all functions  $f[N]$  form the set  $A(f)$ :

$$(4.3) \quad A(f) := \{f[N]\}, \quad N = 0, \dots, 15.$$

The automorphisms  $S^A$  on  $A(f)$  follow directly from the construction of the  $t_{\mathbb{O}[N]}$  above, as the group of repeat application of the  $\mathcal{T}_n$  and identity (id) on the associative triplets  $t_{\mathbb{O}[N]}$ :

$$(4.4) \quad S^A : A(f) \rightarrow A(f),$$

$$(4.5) \quad S^A \cong \{(\text{id}), \mathcal{T}_0\} \times \{(\text{id}), \mathcal{T}_1\} \times \{(\text{id}), \mathcal{T}_2\} \times \{(\text{id}), \mathcal{T}_3\} = \mathbb{Z}_2^4.$$

Distances  $B(f)$  are now constructed from linear superposition of the 16 functions in  $A(f)$ . Left- and right-octonions (with  $N < 8$  and  $N \geq 8$  respectively) will not be treated separately as in [14], instead they will be handled as one set here. A sign matrix is defined:

$$(4.6) \quad b_{jk} := (-1)^{j \wedge k}, \quad j, k = 0, \dots, 15,$$

where  $j \wedge k$  is logical "and" from bitwise representation of the  $j$  and  $k$ . The distances  $B(f)$  then are the Hadamard transforms  $H_4$  on the functions:

$$(4.7) \quad g[k] := \frac{1}{4} \sum_{j=0}^{15} b_{jk} f[j], \quad k = 0, \dots, 15,$$

$$(4.8) \quad B(f) := \{g[k]\} = \left\{ \frac{1}{4} \sum_{j=0}^{15} b_{jk} f[j] \right\} = H_4(f[N]).$$

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<sup>1</sup>Except of course for the trivial case where no octonion multiplication occurred at all, such as e.g.  $f(a_0, a_1) = a_0 + a_1$ .

Since

$$\begin{aligned}
 (4.9) \quad \frac{1}{4} \sum_{k=0}^{15} b_{kl} g[k] &= \frac{1}{16} \sum_{j,k=0}^{15} b_{jk} b_{kl} f[j] = \frac{1}{16} \sum_{j,k=0}^{15} (-1)^{j \wedge k} (-1)^{k \wedge l} f[j] \\
 &= \frac{1}{16} \sum_{j,k=0}^{15} (-1)^{k \wedge (j+l)} f[j] = \frac{1}{16} \sum_{j,k=0}^{15} b_{k(j+l)} f[j] = f[l].
 \end{aligned}$$

the sets of distances  $B(f)$  and functions  $A(f)$  are dual to each other, related through linear superposition using the sign matrix  $b_{jk}$ . This dualism is a property of the Hadamard transform in general, as it is its own inverse.

Rows and columns in the  $b_{jk}$  correspond to functions  $f[k]$  and distances  $g[j]$ . Set set of rows  $\{b_{jk}, k \text{ fixed}\}$  can be constructed from a 16 element vector  $b_{j0}$  and four duality morphisms  $\{\mathcal{T}_n^b\}$  acting on the sign of  $b_{j0}$ :

$$(4.10) \quad b_{j0} = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1),$$

$$(4.11) \quad \mathcal{T}_0^b := (+, +, +, +, +, +, +, +, -, -, -, -, -, -, -, -),$$

$$(4.12) \quad \mathcal{T}_1^b := (+, +, +, +, -, -, -, -, +, +, +, +, -, -, -, -),$$

$$(4.13) \quad \mathcal{T}_2^b := (+, +, -, -, +, +, -, -, +, +, -, -, +, +, -, -),$$

$$(4.14) \quad \mathcal{T}_3^b := (+, -, +, -, +, -, +, -, +, -, +, -, +, -, +, -).$$

All unique combinations of repeat application of the  $\{\mathcal{T}_0^b\}$  and identity (id) on  $b_{j0}$  generate the set of rows  $\{b_{jk}, k \text{ fixed}\}$  (and since  $b_{jk} = b_{kj}$  the similar reasoning applies to the columns  $\{b_{jk}, j \text{ fixed}\}$ ). Given a set of functions  $\{f[N]\}$ , the automorphisms  $S^B$  on the distances  $B(f)$  then are:

$$(4.15) \quad S^B : B(f) \rightarrow B(f),$$

$$(4.16) \quad S^B \cong \{(\text{id}), \mathcal{T}_0^b\} \times \{(\text{id}), \mathcal{T}_1^b\} \times \{(\text{id}), \mathcal{T}_2^b\} \times \{(\text{id}), \mathcal{T}_3^b\} = \mathbb{Z}_2^4.$$

This makes the automorphisms  $S^A$  on the octonion functions  $A(f)$  similar to the automorphisms on the distances  $B(f)$ ,

$$(4.17) \quad S^A \sim S^B.$$

With the definitions from above, the "octonion variance sieve" from [14] is equivalent to computing the  $g[j]$ . An "algebraic invariant" is then built from functions  $f[0]$  such that:

$$(4.18) \quad g[j] = 0 \text{ for } j > 0.$$

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#### GLOSSARY OF SYMBOLS

$b_{\mathbb{O}}$	An ordered set of orthogonal vectors $\{1, i_1, \dots, i_7\} \in \mathbb{R}^8$ .
$\mathbb{O}[N]$	An octonion algebra with a given a $b_{\mathbb{O}}$ as basis and one of 16 multiplication rules indexed with $N = 0, \dots, 15$ .
$t_{\mathbb{O}[N]}$	The seven associative triplets of basis elements from $\mathbb{O}[N]$ .

$f[N]$	Polynomial functions on $\mathbb{R}^8$ , $f[N] : \mathbb{R}^8 \times \dots \times \mathbb{R}^8 \rightarrow \mathbb{R}^8$ , that use $\odot[N]$ for multiplication.
$A(f)$	Given a polynomial $f$ , the set of functions $A(f) := \{f[N]\}$ , $N = 0, \dots, 15$ .
$\mathcal{T}_0, \dots, \mathcal{T}_3$	Duality automorphisms on the multiplication rules, $\mathcal{T}_n : \{\odot[N]\} \rightarrow \{\odot[N]\}$ , $\mathcal{T}_n \mathcal{T}_n = (\text{id})$ (for $n = 0, 1, 2, 3$ ).
$S^A$	The automorphisms on $A(f)$ , i.e.: $S^A : A(f) \rightarrow A(f)$ .
$b_{jk}$	A $16 \times 16$ sign matrix, $b_{jk} := (-1)^{j \wedge k}$ ; $j, k = 0, \dots, 15$ .
$H_4$	Hamadard transform generated by the $b_{jk}$ .
$g[N]$	A "distance" function obtained from linear superposition of the $f[N]$ , using $b_{jk}$ as coefficients.
$B(f)$	Given a polynomial $f$ , the set of distance functions $B(f) := H_4(A(f)) = \{g[N]\}$ , $N = 0, \dots, 15$ .
$\mathcal{T}_0^b, \dots, \mathcal{T}_3^b$	Duality automorphisms on the rows (or columns) of $b_{jk}$ .
$S^B$	The automorphisms on $B(f)$ , i.e.: $S^B : B(f) \rightarrow B(f)$ .

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