An order on the Leech lattice from a $\mathbb{Z}_3$-symmetric triple-octonion product

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Abstract

We show that the Leech lattice Λ admits the structure of an order in a 24-dimensional real algebra. The product is a $\mathbb{Z}_3$-symmetric triple-octonion product on $\mathbb{R}^{24}$, assembled from nine octonion-multiplication blocks under cyclic $\mathbb{Z}_3$ permutation of three octonion factors. In Robert A. Wilson's sublattice description of Λ [J. Algebra **322** (2009), 2186–2190], closure turns on a transposition σ of two imaginary basis units: σ leaves the E_8 lattice invariant and is an algebra isomorphism of the octonions, but it moves Wilson's sublattices $L\bar{s}$ and Ls. Reduced modulo $2L$, the construction produces a Witt decomposition of $L/2L$ into a complementary pair of Lagrangians, opening the natural research program of which polarizations of $L/2L$ lift to a closed product on Λ.

We further provide a historical account from the original sources (1923–1946) in Appendix B: Leonard E. Dickson gave the first verified maximal order of integral octonions (1923), with an undercount of the total; Johannes Kirmse (1924) recovered the full count of seven but exhibited one non-closed candidate; a transposition provided by Richard H. Bruck repaired that candidate (1946), with H. S. M. ("Donald") Coxeter providing the first correct account of all seven maximal orders. The map σ of the present construction is exactly this transposition, acting on a different sublattice.

This result was developed through a systematic human–AI collaboration; the complete research record is publicly available [repo].

1 Introduction

The Leech lattice Λ is the unique even unimodular lattice of rank 24 with no vectors of squared norm 2 [ConwaySloane1999]. Its minimal shell $\mathrm{Min}(\Lambda)$ consists of 196,560 vectors of squared norm 8, and its automorphism group is the Conway group Co_0, a double cover of the sporadic simple group Co_1.

In rank 8, the E_8 lattice L carries an octonionic multiplication that makes it a maximal order in this algebra. Seven octonion-multiplication conventions on $\mathbb{R}^8$ have this property; the resulting structures are in bijection with the seven maximal orders of the integral octonions [Coxeter1946, ConwaySmith2003]. Wilson [Wilson2009, Section 3] showed that Λ embeds naturally in L^3 via three sublattice conditions, raising the question: does Λ inherit an algebraic structure from the octonion product?

This question has been studied from several angles. Geoffrey M. Dixon [Dixon2010] constructed Λ using an octonionic XY-product. John C. Baez and Greg Egan [Baez2014] explored Jordan-algebraic structures on Λ within the exceptional Jordan algebra $\mathfrak{h}_3(\mathbb{O})$, finding closure under a doubled Jordan product.

The $\mathbb{Z}_3$-symmetric triple-octonion product on L^3 – using the same octonion multiplication in all three blocks with $\mathbb{Z}_3$-symmetric cross-block routing – satisfies two of Wilson's three sublattice conditions automatically. The third, however, is generally not closed under octonion products, including Wilson's.

The main result of this paper is that closure on Λ is restored by twisting the octonion multiplication on each block by a transposition σ of two imaginary basis units. The map σ leaves the E_8 lattice L invariant and is an algebra isomorphism on the octonions, but it moves Wilson's sublattices such that inclusion holds.

Theorem 1.1. *Let $\star$ denote the $\mathbb{Z}_3$-symmetric triple-octonion product on $\mathbb{R}^{24} = \mathbb{O}_1 \oplus \mathbb{O}_2 \oplus \mathbb{O}_3$ under $\mathbb{Z}_3$ cross-block routing (Definition 3.3). Then $u \star v \in \Lambda$ for all $u, v \in \Lambda$ in Wilson's representation [Wilson2009].*

Corollary 1.2. $(\Lambda, +, \star)$ *is an order*[1] *in the $\mathbb{R}$-algebra* $(\mathbb{R}^{24}, +, \star)$.

Proof. The product $\star$ is bilinear (it is a sum of bilinear octonion products). By Theorem 1.1, $\star$ maps $\Lambda \times \Lambda$ into Λ. ☐

Remark 1.3 (Order-closure, not shell-closure). The closure statement $\Lambda\star\Lambda \subseteq \Lambda$ means that products land somewhere in Λ; it does not assert that the minimal shell is preserved. Empirically, the product norm on $\mathrm{Min}(\Lambda) \times \mathrm{Min}(\Lambda)$ takes values in $\{16, 32, 48, 64, 80, 96, 112, 128\}$ (see Section 5), none of which equals the minimal value $N = 8$: products of minimal vectors always land on strictly higher shells. This is consistent with $(\Lambda, +, \star)$ being an order in the ring-theoretic sense – a subring that is a free $\mathbb{Z}$-module of rank equal to the ambient algebra's dimension – a property weaker than, and independent of, any shell-level closure.

2 Preliminaries

2.1 The octonion algebra

Let $\mathbb{O}$ denote the real octonion algebra with basis $\{e_0, e_1, \ldots, e_7\}$, where e_0 is the identity. The multiplication is determined by the seven *Fano triples*

$$(1,2,4),\ (2,3,5),\ (3,4,6),\ (4,5,7),\ (5,6,1),\ (6,7,2),\ (7,1,3), \tag{2.1}$$

encoding the rule: for each triple (a, b, c), $e_a \cdot e_b = +e_c$, with cyclic permutations carrying a positive sign and anti-cyclic permutations carrying a negative sign [Baez2002]. Every imaginary unit squares to $-e_0$. Among *unital* real algebras, $\mathbb{O}$ is the unique normed division algebra of dimension 8 (Hurwitz); relaxing "division algebra" to the absence of zero divisors (equivalently, bijectivity of left and right multiplication by every nonzero element) admits further examples, including the non-unital Okubo algebra.

The *norm form* is $N(x) = x\bar{x} = \sum_{i=0}^{7} x_i^2$ for $x = \sum_i x_i e_i$, where $\bar{x} = x_0 e_0 - \sum_{i=1}^{7} x_i e_i$ is the conjugate. The composition property $N(xy) = N(x)N(y)$ holds.

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[1] The ambient algebra has no multiplicative identity (Section 5), so the term "order" is used here in the non-unital sense.

2.2 The E_8 lattice

In Wilson's setup [Wilson2009], the E_8 lattice is $L = D_8^+$, the set of vectors in $\mathbb{R}^8$ whose coordinates (in the basis $\{e_0, \dots, e_7\}$) are either all integers with even coordinate sum, or all half-integers with odd coordinate sum [ConwaySmith2003]. It has 240 roots (vectors of squared norm 2). Under the standard octonion multiplication, L is a maximal order in $\mathbb{O}$: it is closed under multiplication and admits no enlargement to a larger order [Coxeter1946, ConwaySmith2003]; for a modern survey of the integral-octonion history, see Petersson [Petersson2018].

2.3 Wilson's Leech lattice

Define $s = \frac{1}{2}(-e_0 + e_1 + \cdots + e_7)$ and $\bar{s} = \frac{1}{2}(-e_0 - e_1 - \cdots - e_7)$, both of norm $N(s) = N(\bar{s}) = 2$. Then $s \in L$ (it is a type-2 root, with an odd coordinate sum), while $\bar{s} \notin L$ (its coordinate sum is -4, which falls in the excluded even branch of the half-integer case). Let

$$L\bar{s} = \{\ell \cdot \bar{s} : \ell \in L\}, \qquad Ls = \{\ell \cdot s : \ell \in L\}.$$

Both $L\bar{s}$ and Ls are sublattices of L,[2] with $[L : L\bar{s}] = [L : Ls] = 16$, $L\bar{s} + Ls = L$, and $L\bar{s} \cap Ls = 2L$.

Definition 2.1 (Wilson)**.** The *Leech lattice* is

$$\Lambda = \bigl\{(x, y, z) \in L^3 : (1)\ x, y, z \in L;\ (2)\ x{+}y,\ x{+}z,\ y{+}z \in L\bar{s};\ (3)\ x{+}y{+}z \in Ls\bigr\}.$$

The 196,560 minimal vectors of Λ decompose into three families:

Family	Formula	Count
Type 1	$(2\lambda, 0, 0)$ and cyclic permutations	$3 \times 240 = 720$
Type 2	$(\lambda\bar{s},\ (\lambda\bar{s})j,\ 0)$ and cyclic perms	$3 \times 240 \times 16 = 11{,}520$
Type 3	$((\lambda s)j,\ \lambda k,\ (\lambda j)k)$ and cyclic perms	$3 \times 240 \times 16^2 = 184{,}320$

Here λ ranges over the 240 roots of L, and $j, k \in J = \{\pm e_t : 0 \le t \le 7\}$ (16 unit octonions).

3 The construction

Definition 3.1 (Transposition)**.** Let $\sigma = (i\ j)$ be a transposition on $\{1, \dots, 7\}$, extended to the linear involution $\sigma\colon \mathbb{R}^8 \to \mathbb{R}^8$ that fixes e_0 and swaps $e_i \leftrightarrow e_j$.

Definition 3.2 (σ-twisted octonion product)**.** The *σ-twisted octonion product* $\cdot_\sigma$ on $\mathbb{O}$ is the σ-twisted variant of the standard octonion product (2.1):

$$x \cdot_\sigma y \ := \ \sigma\bigl(\sigma(x) \cdot \sigma(y)\bigr). \tag{3.1}$$

This is another octonion product on $\mathbb{O}$: by definition, $\sigma\colon (\mathbb{O}, \cdot) \to (\mathbb{O}, \cdot_\sigma)$ is an algebra isomorphism, $\sigma(x \cdot y) = \sigma(x) \cdot_\sigma \sigma(y)$.

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[2] $L\bar{s} \subseteq L$ even though $\bar{s} \notin L$: each $\ell \cdot \bar{s}$ with ℓ a root of L lies in L by direct computation, and the inclusion extends by $\mathbb{Z}$-linearity in the left factor. $L\bar{s}$ is therefore a $\mathbb{Z}$-subgroup of L, and is moreover closed under left-multiplication by L (Wilson's condition (2) for the untwisted product, implicit in the sublattice argument). For Ls, the subgroup property is likewise immediate from $s \in L$. Unlike $L\bar{s}$, the sublattice Ls is not closed under the octonion product itself ($Ls \cdot Ls \not\subseteq Ls$) – and that failure is precisely the obstruction resolved in this paper (Remark 4.5).

Definition 3.3 (Triple product with $\mathbb{Z}_3$ routing)**.** Decompose $\mathbb{R}^{24} = \mathbb{O}_1 \oplus \mathbb{O}_2 \oplus \mathbb{O}_3$ into three copies of $\mathbb{R}^8$ (blocks indexed $\alpha = 1, 2, 3$). For $u = (x, y, z)$ and $v = (x', y', z')$ with $x, x' \in \mathbb{O}_1$, etc., define the *triple product*

$$u \star v = (P_1,\, P_2,\, P_3), \tag{3.2}$$

where each block P_α is the sum of three σ-twisted octonion products:

$$P_1 = x \cdot_\sigma x' + z \cdot_\sigma y' + y \cdot_\sigma z', \tag{3.3}$$
$$P_2 = y \cdot_\sigma y' + x \cdot_\sigma z' + z \cdot_\sigma x', \tag{3.4}$$
$$P_3 = z \cdot_\sigma z' + y \cdot_\sigma x' + x \cdot_\sigma y'. \tag{3.5}$$

The routing rule is: a product of blocks α and β lands in block γ, where $\{\alpha, \beta, \gamma\} = \{1, 2, 3\}$ for $\alpha \neq \beta$, and in block α when $\alpha = \beta$. All nine block-pair products use the same multiplication $\cdot_\sigma$.

4 Proof of closure

We prove Theorem 1.1 by showing that the σ-twisted octonion product preserves each of Wilson's three conditions (Definition 2.1). From the given choice of octonion basis and Fano triples, the load-bearing effect of σ is that it *moves* the sublattices $L\bar{s}$ and Ls – this is where the work of the proof lives – while leaving the octonions and L themselves invariant.

4.1 The four lemmas

Lemma 4.1. *$\sigma(L) = L$.*

Proof. The lattice $L = D_8^+$ consists of all vectors in $\mathbb{R}^8$ whose coordinates are either all integers with even coordinate sum, or all half-integers with odd coordinate sum (Section 2). The transposition σ permutes two coordinates, which preserves both the integer/half-integer parity and the coordinate sum. Hence $\sigma(L) \subseteq L$. Since σ is an involution, $\sigma(L) = L$. $\square$

For each of the next three lemmas, L, $\sigma(L\bar{s})$, and $\sigma(Ls)$ are full-rank $\mathbb{Z}$-lattices of rank 8 in $\mathbb{R}^8$. By $\mathbb{Z}$-bilinearity of the octonion product, an inclusion $A \cdot B \subseteq C$ between such lattices holds if and only if it holds on a chosen pair of $\mathbb{Z}$-bases: if $\{a_i\}_{i=1}^8$ generates A and $\{b_j\}_{j=1}^8$ generates B, then $A \cdot B \subseteq C$ is equivalent to each of the 64 products $a_i \cdot b_j$ lying in $\mathbb{Z}\langle c_1, \ldots, c_8\rangle$ for a chosen $\mathbb{Z}$-basis $\{c_k\}_{k=1}^8$ of C. (The forward direction is the substantive one and uses bilinearity; the reverse direction is immediate since each $a_i \in A$ and each $b_j \in B$.) Each lemma below is proved by exhibiting that finite list of integer coefficients explicitly. Concrete $\mathbb{Z}$-bases for L, $\sigma(Ls)$, and $\sigma(L\bar{s})$ are fixed in Appendix A (denoted there L_k, M_k, N_k respectively); the 64-entry coefficient tables for the three lemmas appear as Tables 1, 2, and 3.

Lemma 4.2. *$L \cdot L \subseteq L$.*

Proof. The 64 products $L_i \cdot L_j$ are integer combinations of $\{L_1, \ldots, L_8\}$; the explicit coefficients are Table 1. By $\mathbb{Z}$-bilinearity, $L \cdot L \subseteq L$. This recovers the classical fact that $L = D_8^+$ is a maximal order in $\mathbb{O}$ [Coxeter1946, ConwaySmith2003]. $\square$

Lemma 4.3. *$L \cdot \sigma(L\bar{s}) \subseteq \sigma(L\bar{s})$.*

Proof. The 64 products $L_i \cdot N_j$ are integer combinations of $\{N_1, \dots, N_8\}$; the explicit coefficients are Table 2. By $\mathbb{Z}$-bilinearity, $L \cdot \sigma(L\bar{s}) \subseteq \sigma(L\bar{s})$. □

Lemma 4.4. *$\sigma(Ls) \cdot \sigma(Ls) \subseteq \sigma(Ls)$.*

Proof. The 64 products $M_i \cdot M_j$ are integer combinations of $\{M_1, \dots, M_8\}$; the explicit coefficients are Table 3. By $\mathbb{Z}$-bilinearity, $\sigma(Ls) \cdot \sigma(Ls) \subseteq \sigma(Ls)$. □

Remark 4.5. The sublattice Ls is *not* closed under the standard octonion product, and $\sigma(Ls) \neq Ls$: with $\sigma = (1\ 2)$, the element $v = (-1, 0, 1, 0, 0, 1, 0, 1)$ lies in $\sigma(Ls) \setminus Ls$. The twist therefore moves the block-sum sublattice off Ls onto the closed sublattice $\sigma(Ls)$ (Lemma 4.4). Likewise $\sigma(L\bar{s}) \neq L\bar{s}$: $w = (0, 0, -1, 0, -1, 0, -1, -1) \in \sigma(L\bar{s}) \setminus L\bar{s}$.

4.2 Blockwise form of the triple product

Since each σ-twisted product has the form $a \cdot_\sigma b = \sigma(\sigma(a) \cdot \sigma(b))$ and σ is linear, each block P_α is σ applied to a sum of standard octonion products of σ-images of the inputs. Write $X = \sigma(x)$, $Y = \sigma(y)$, $Z = \sigma(z)$, $X' = \sigma(x')$, $Y' = \sigma(y')$, $Z' = \sigma(z')$:

$$P_1 = \sigma\big(X \cdot X' + Z \cdot Y' + Y \cdot Z'\big), \tag{4.1}$$

$$P_2 = \sigma\big(Y \cdot Y' + X \cdot Z' + Z \cdot X'\big), \tag{4.2}$$

$$P_3 = \sigma\big(Z \cdot Z' + Y \cdot X' + X \cdot Y'\big). \tag{4.3}$$

Taking sums:

- **Block sum.** $P_1 + P_2 + P_3 = \sigma\big((X + Y + Z) \cdot (X' + Y' + Z')\big)$.
- **Pairwise sums.** For example, $P_1 + P_2 = \sigma\big(X \cdot (X' + Z') + Y \cdot (Y' + Z') + Z \cdot (X' + Y')\big)$, and similarly for the other two pairs.

4.3 Wilson's condition (1): each block lies in L

Proposition 4.6. *For all $u, v \in \Lambda$, each block P_α of $u \star v$ lies in L.*

Proof. Consider $P_1 = \sigma(X \cdot X' + Z \cdot Y' + Y \cdot Z')$. Since $u \in \Lambda$, condition (1) gives $x \in L$, so $X = \sigma(x) \in L$ by Lemma 4.1. Similarly $X', Y, Y', Z, Z' \in L$. By Lemma 4.2, each product $X \cdot X'$, $Z \cdot Y'$, $Y \cdot Z'$ lies in L. Their sum lies in L (closed under addition). Applying σ and using Lemma 4.1, $P_1 \in L$. The same argument applies to P_2 and P_3. □

4.4 Condition (2): pairwise sums lie in $L\bar{s}$

Proposition 4.7. *For all $u, v \in \Lambda$, the pairwise sums $P_\alpha + P_\beta$ lie in $L\bar{s}$.*

Proof. We show $P_1 + P_2 \in L\bar{s}$; the other two cases follow by the $\mathbb{Z}_3$ symmetry of the routing.

From (4.1)–(4.2):

$$P_1 + P_2 = \sigma\big(X \cdot (X' + Z') + Y \cdot (Y' + Z') + Z \cdot (X' + Y')\big).$$

Since $v \in \Lambda$, condition (2) gives $x' + z' \in L\bar{s}$, so $X' + Z' = \sigma(x' + z') \in \sigma(L\bar{s})$. Similarly $Y' + Z' \in \sigma(L\bar{s})$ and $X' + Y' \in \sigma(L\bar{s})$. Since $u \in \Lambda$, we have $X, Y, Z \in L$ (by Lemma 4.1).

By Lemma 4.3, each product $X \cdot (X' + Z')$, $Y \cdot (Y' + Z')$, $Z \cdot (X' + Y')$ lies in $\sigma(L\bar{s})$. Their sum lies in $\sigma(L\bar{s})$. Applying the outer σ:

$$P_1 + P_2 = \sigma(\text{element of } \sigma(L\bar{s})) \in \sigma(\sigma(L\bar{s})) = L\bar{s},$$

since $\sigma^2 = \mathrm{id}$. ☐

4.5 Condition (3): the block sum lies in Ls

This is the crux of the proof. For the standard (untwisted) product the block sum would require $Ls{\cdot}Ls \subseteq Ls$, which does not hold (Remark 4.5). The twist redirects the computation through $\sigma(Ls)$, where closure holds.

Proposition 4.8. *For all $u, v \in \Lambda$, the block sum $P_1 + P_2 + P_3$ lies in Ls.*

Proof. From (4.1)–(4.3):

$$P_1 + P_2 + P_3 = \sigma\big((X + Y + Z) \cdot (X' + Y' + Z')\big).$$

Since $u \in \Lambda$, condition (3) gives $x + y + z \in Ls$, so

$$X + Y + Z = \sigma(x + y + z) \in \sigma(Ls).$$

Similarly $X' + Y' + Z' \in \sigma(Ls)$.

By Lemma 4.4, $\sigma(Ls) \cdot \sigma(Ls) \subseteq \sigma(Ls)$, so

$$(X + Y + Z) \cdot (X' + Y' + Z') \in \sigma(Ls).$$

Applying the outer σ:

$$P_1 + P_2 + P_3 \in \sigma(\sigma(Ls)) = Ls.$$ ☐

This completes the proof of Theorem 1.1.

Remark 4.9 (Consecutive twists). The construction extends to consecutive twists $\sigma_2 \circ \sigma_1$ by another transposition. The composite acts as a single twist by $\pi = \sigma_2\sigma_1$; as a product of two transpositions, π is the identity, a 3-cycle, or a $(2, 2)$-double transposition. Tested by exact-arithmetic enumeration on a $\mathbb{Z}$-basis of Λ (`python_project/src/verify_consecutive_twists_exact.py`): π = identity does *not* close (the untwisted product); of the 70 3-cycles, exactly half close – for each three-element subset of imaginary indices, one of the two cyclic orientations does; of the 105 $(2, 2)$'s, the 21 whose fixed three indices form a Fano line all fail, and of the remaining 84 exactly half close (42 in all).

5 Algebraic properties

The order $(\Lambda, +, \star)$ has been tested for several standard algebraic identities. Samples are drawn uniformly (by family weight) from $\mathrm{Min}(\Lambda) \times \mathrm{Min}(\Lambda)$ for the binary identities, and from $\mathrm{Min}(\Lambda)$ for the cube and quartic identities. Sample size is $N = 10^6$ per property; exact integer arithmetic throughout. The standard error is $\mathrm{SE} = \sqrt{p(1-p)/N}$, where p is the fraction of samples on which the identity holds.

Property	Holds?	Fraction of samples (% ± SE)
Multiplicative identity	No	n/a (structural)
Commutativity $u \star v = v \star u$	No	$(0.018 \pm 0.0014)\%$
Norm multiplicativity $N(u \star v) = N(u)N(v)$	No	$(46.77 \pm 0.050)\%$
Left alternativity $(x \star x) \star y = x \star (x \star y)$	No	$(0.188 \pm 0.0043)\%$
Right alternativity $(x \star y) \star y = x \star (y \star y)$	No	$(0.188 \pm 0.0043)\%$
Flexibility $(x \star y) \star x = x \star (y \star x)$	No	$(8.37 \pm 0.028)\%$
Cube power-associativity $x^2 \star x = x \star x^2$	No	$(30.70 \pm 0.046)\%$
Quartic power-associativity	No	$(12.75 \pm 0.033)\%$
Symmetric composition $\langle u \star v, w \rangle = \langle u, v \star w \rangle$	No	$(25.37 \pm 0.044)\%$

The failure of norm multiplicativity rules out $(\Lambda, +, \star)$ being a composition algebra; the failure of symmetric composition rules out, independently, its being a symmetric composition algebra in the sense of Okubo and para-Hurwitz (Alberto Elduque [Elduque1996, Elduque2023_IsotropicNorm]; Susumu Okubo [Okubo1995_Book]). Cube and quartic power-associativity are reported separately, the quartic identity being strictly stronger: in a joint exact-arithmetic test on 3×10^5 samples, every quartic-passing sample also satisfied the cube identity, while the cube-passing set was roughly 2.4× larger.

Remark 5.1. The product norm on $\mathrm{Min}(\Lambda) \times \mathrm{Min}(\Lambda)$ takes values in $\{16, 32, 48, 64, 80, 96, 112, 128\}$, all multiples of 16. The mode is 64 (the norm-multiplicative value $N(u)N(v) = 8 \times 8$).

Remark 5.2. The ambient algebra $(\mathbb{R}^{24}, +, \star)$ has no multiplicative identity: any putative left identity (a, b, c) would have to act on an arbitrary (x', y', z') by the block matrix $\left(\begin{smallmatrix} a & c & b \\ c & b & a \\ b & a & c \end{smallmatrix}\right)$ (entries acting by left-multiplication in $\mathbb{O}$), and no triple (a, b, c) makes this the identity on all three blocks simultaneously. In particular, (e_0, e_0, e_0) is not an identity: it maps (x', y', z') to $(x' + y' + z',\ x' + y' + z',\ x' + y' + z')$. Consequently $(\Lambda, +, \star)$ is a non-unital order.

The order does not satisfy any of the classical algebraic identities (associativity, alternativity, flexibility, power-associativity, commutativity). The defining property is *closure*: the Leech lattice is stable under the product.

6 Related work

Wilson (2009) [Wilson2009] characterises Λ as $\{(x, y, z) \in L^3 : \text{conditions 1–3}\}$, gives an explicit listing of the 196,560 minimal vectors in the form $3 \times 240 \times (1 + 16 + 16 \times 16)$, and proves that Λ is even and self-dual. The sublattice framework is the setting used in the present paper.

Dixon (2010) [Dixon2010] constructs Λ using the octonion XY-product[3]

$$A \circ_{XY} B \;=\; (AX)(Y^{\dagger}B),$$

where $X, Y \in \mathbb{O}$ are chosen so that $YX^{\dagger}$ is a unit octonion (the identity of the new multiplication). When $Y^{\dagger} = X^{-1}$ the XY-product specialises to the X-product $A \circ_X B = (AX)(X^{-1}B)$; Dixon uses

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[3] The construction has an earlier precursor in Dixon [Dixon1995], where a closely related octonionic description of Λ is proposed. The formula there required a small correction in the third family of minimal vectors, which Wilson [Wilson2009, end of §5] supplied in a footnote-style remark while independently deriving the sublattice characterisation. The situation is similar to the Kirmse–Bruck–Coxeter relationship recorded in Appendix B: an early, structurally correct proposal needing correction.

this X-product with $X = \ell_0 = \frac{1}{2}(1+e_1+\cdots+e_7)$ to obtain E_8, and general XY-products for higher-dimensional laminated lattices including Λ. We have independently reproduced this construction in the supplemental code [repo].

Baez and Egan (2014) [Baez2014] study closure properties of integer sublattices of the exceptional Jordan algebra $\mathfrak{h}_3(\mathbb{O})$. In parts 9–10 of the series, Baez considers the 27-dimensional lattice $\mathbb{Z}^3 \oplus L_L \subset \mathfrak{h}_3(\mathbb{O})$, with integers on the diagonal and an off-diagonal Leech sublattice $L_L \cong \Lambda$ embedded as an octonionic triple. This lattice is *not* closed under the Jordan product $x \circ y = xy + yx$, but *is* closed under the doubled product $2(xy + yx)$. Egan further enumerated at least 17,280 distinct off-diagonal Leech embeddings $L_L \subset \mathbb{O}^3$ for which the associated 27-dimensional lattice is closed under this doubled product (see part 10 of [Baez2014] for Egan's computational counts). Baez left open the question whether any such lattice is closed under the undoubled Jordan product – which would remove the factor-of-2 discrepancy – and conjectures that the answer is negative. We have independently reproduced the full Baez–Egan verification chain – the E_8 simple-roots data, the lemma $L_1 \cap L_2 = 2L_0$, the Leech-lattice theorem, the doubled-Jordan-product closure, the derivation of the lower bound 244,035,421 on the number of E_8^3-orbit Leech embeddings, and Egan's sharper count of 17,280 Jordan rings. This last factors as $17{,}280 = 270 \times 64$: the 270 is the number of sublattices $L_1 \subset L$ isometric to $\sqrt{2}\,L$, equivalently the number of 4-dimensional maximal totally isotropic subspaces of $L/2L \cong \mathbb{F}_2^8$ under the plus-type quadratic form induced by the E_8 Gram form; the 64 is the number of L_2 in this collection that meet a given L_1 only in $2L$, equivalently the number of 4-dimensional complementary isotropic subspaces. All Baez–Egan claims hold under our reading. Details and code are in [repo].

Comparison. The product $\star$ is bilinear on $\mathbb{R}^{24}$ itself. It does not require an enclosing 27-dimensional Jordan-algebraic ambient within which the Leech sits as a subspace (cf. [Baez2014]), and the cross-block S_3 extending the $\mathbb{Z}_3$ routing is supplied by the routing of Definition 3.3 rather than by the octonion product itself: Dixon's XY-product [Dixon1995, Dixon2010] is a useful device for describing each E_8 block, but it is not in itself $\mathbb{Z}_3$-symmetric across the three factors. Whether the Leech embedding $\sigma(\Lambda) \subset \mathbb{O}^3$ obtained by applying σ block-wise overlaps with one of the embeddings catalogued by Egan, or sits outside that family, is a question this paper does not address.

Anatomy of the Baez–Egan closure on Λ. Egan and Baez's closure is on the ambient 27-dimensional lattice $\widetilde{\Lambda} = \mathbb{Z}^3 \oplus \Lambda \subset \mathfrak{h}_3(\mathbb{O})$, not on Λ alone. For off-diagonal Hermitian matrices $X, Y \in \mathfrak{h}_3(\mathbb{O})$ representing $u = (x, y, z)$ and $v = (x', y', z')$ in $\Lambda \subset \mathbb{O}^3$, direct computation gives

$$2(XY + YX) \;=\; D \;+\; R,$$

where R is an off-diagonal residue lying in Λ and $D \in \mathbb{Z}^3$ is a diagonal piece whose entries are four times the block-pair Euclidean inner products of u and v:

$$D \;=\; 4\big(\langle x, x'\rangle + \langle y, y'\rangle,\ \langle x, x'\rangle + \langle z, z'\rangle,\ \langle y, y'\rangle + \langle z, z'\rangle\big),$$

so $\operatorname{tr} D = 8\langle u, v\rangle$ recovers the $\mathbb{R}^{24}$ inner product of u and v. Projecting onto the off-diagonal subspace defines a $\mathbb{Z}$-bilinear closed product $\varphi(u, v) := R$ on Λ – an order in the sense of Theorem 1.1, but obtained by projecting from a larger ambient with the inner-product shadow D discarded. For minimal Leech vectors $u \in \operatorname{Min}(\Lambda)$, squaring gives $\operatorname{tr} D = 8N(u) = 64$, so the projection is non-trivial on every minimal-vector input. The induced product φ is moreover commutative (the Jordan product is symmetric in its arguments), whereas $\star$ is not (Section 5); the two orders are therefore distinct bilinear products on $\mathbb{R}^{24}$ that both close on Λ, reaching that closure through different ambient algebras.

Earlier precursors. Further back, several constructions foreshadow the octonion-triple and Jordan-algebra frameworks above. John H. Conway and Neil J. A. Sloane [ConwaySloane1982] give 23 constructions of Λ, one for each Niemeier lattice, including a Turyn-style construction from three copies of E_8. James Lepowsky and Arne Meurman [LepowskyMeurman1982] describe Λ as a sublattice of E_8^3 containing $2E_8^3$ and use this to generate the Conway group Co_0 via permutations of the three E_8 factors together with one extra automorphism. Noam D. Elkies and Benedict H. Gross [ElkiesGross1996] obtain Λ from an integer form of the exceptional Jordan algebra $\mathfrak{h}_3(\mathbb{O})$ as the orthogonal complement of a distinguished identity element, and show that the automorphism group of this integer form is generated by octonion reflections [*octave reflections*, in their terminology]. Each of these constructions is additive or geometric: none defines or tests a bilinear product on Λ.

Leonard E. Dickson (1923), Kirmse (1924), Kurt Mahler (1942), Bruck and Coxeter (1946). For a full historical account, see Appendix B.

Daniele Corradetti (2026). A thematically adjacent recent construction is Corradetti's order on E_8 via the Okubo algebra [Corradetti2026]: a different non-classical bilinear structure on E_8, obtained by a 2-adic conductor / saturation of an Okubo-algebra overlattice. The two operations involved differ from the present mod-2 quotient (Appendix A, Section A.1) – Corradetti rescales an order and 2-adically saturates a sublattice; the present paper passes to a quotient $L \to L/2L$ – but both place 2-primary structure at the centre of an octonion-derived order on E_8.

7 Conclusion

The Leech lattice admits an order under a remarkably simple construction: a $\mathbb{Z}_3$-symmetric triple-octonion product on $\mathbb{R}^{24}$, realised in the Coxeter representation aligned with Wilson's sublattice description of Λ. The symbolic proof (Section 4) reduces closure to four lemmas, of which the crux is Lemma 4.4: the sublattice $\sigma(Ls)$ – unlike Ls itself – is closed under the standard octonion product. The twist maps the condition-3 sublattice to this closed sublattice, and the involutory nature of σ maps the result back.

Several questions remain immediately open:

- A structural reason for Lemma 4.4. The proof in Appendix A exhibits the 64 integer coefficients certifying $\sigma(Ls) \cdot \sigma(Ls) \subseteq \sigma(Ls)$, but does not explain *why* $\sigma(Ls)$ is closed under the standard octonion product while Ls is not. The mod-2 reduction in Section A.1 identifies what the appendix tables certify after reduction modulo $2L$: $W := \sigma(Ls)/2L$ is a subalgebra of the $\mathbb{F}_2$-octonion-algebra $L/2L$. An integer-level structural reason – an identification of $\sigma(Ls)$ with a left ideal, with the image under a norm-preserving octonion automorphism, or with some module-theoretic structure over L, that would lift the mod-2 picture – would replace the appendix tables by a structural argument. (See the closing paragraphs of Section A.1 for a Lagrangian-polarization reframing of this open question as a research program.)

- The automorphism group of $(\Lambda, +, \star)$ and its relationship to the Conway group Co_0. Empirically, $S_3 \subseteq \mathrm{Aut}(\Lambda, +, \star) \subsetneq \mathrm{Co}_0$: the block-permutation S_3 acts by automorphisms, and the inclusion in Co_0 is strict because $-I_{24} \in \mathrm{Co}_0$ does not preserve $\star$. Full identification of $\mathrm{Aut}(\Lambda, +, \star)$ is open.

- Whether the order is maximal in any appropriate sense.

- Classification of the algebra $(\mathbb{R}^{24}, +, \star)$: it is non-associative, non-alternative, non-flexible, and non-unital (as an order), placing it outside the classical families.

- Whether other linear coordinate permutations of $\mathbb{R}^8$ (beyond simple transpositions of imaginary axes) conjugate the octonion multiplication into a triple product that closes on Λ in the sense of Theorem 1.1.

8 Outlook

A more natural classification of $(\mathbb{R}^{24}, +, \star)$ may emerge from a *ternary* rather than binary viewpoint. The Leech lattice carries a prominent $\mathbb{Z}_3$-symmetry (the cyclic permutation of the three octonion blocks), and $\star$ is assembled from nine bilinear building blocks routed by that $\mathbb{Z}_3$-action. Ternary composition algebras, as studied by Elduque [Elduque1996, Elduque2000_Triality, Elduque2023_IsotropicNorm] and others [SmithVojtechovsky2022, KamiyaOkubo2015, Okubo1995_Book], are typically non-associative, non-commutative, non-alternative, and non-power-associative – the same negative classification $(\mathbb{R}^{24}, +, \star)$ satisfies – yet carry high internal symmetry, with order-3 automorphisms playing the central role: the Okubo algebra has automorphism group SU(3), and its product is built from an order-3 automorphism of the octonions. Jonathan I. Hall's monograph [Hall2019_Moufang] establishes that Moufang loops and groups with triality are essentially the same object. Recasting $\star$ as a trace-like reduction of a ternary operation compatible with the $\mathbb{Z}_3$-block symmetry is a natural direction for further work.

Programmes that model fundamental physics on normed-division-algebra, triality, S_3 symmetry, or symmetric-composition-algebra structures [GresnigtGourlayVarma2023, Koeplinger2023, FureyHughes2025_TrioOfTrialities, MarraniCorradettiZucconi2025] are one setting in which a non-classical bilinear product on $\mathbb{R}^{24}$ may be of interest.

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A Explicit basis tables for Lemmas 4.2–4.4

We fix concrete $\mathbb{Z}$-bases for the lattices L, $\sigma(Ls)$, and $\sigma(L\bar{s})$, and tabulate the integer coefficients of the basis-by-basis products called for in the proofs of Lemmas 4.2, 4.3, and 4.4. Each table has 64 entries and is arranged so that the row labelled (i, j) gives the unique 8-tuple $(c_1, \ldots, c_8) \in \mathbb{Z}^8$ representing the corresponding product in the basis indicated. All entries are integer; closure under the standard octonion product follows by $\mathbb{Z}$-bilinearity. The same identities have been recomputed in two independent computer-algebra implementations as a cross-check (Appendix C).

Coordinates throughout are with respect to the basis $\{e_0, \ldots, e_7\}$ and the standard Fano triples (2.1).

The basis $\{L_k\}$**.** Concretely: $L_1 = s = \frac{1}{2}(-e_0 + e_1 + \cdots + e_7)$, $L_2 = e_0 + e_1$, $L_3 = e_0 - e_1$, and $L_{k+2} = e_0 + e_k$ for $k = 2, \ldots, 6$. The eight vectors are linearly independent over $\mathbb{Q}$ and span L over

$\mathbb{Z}$.

$$\begin{aligned}
L_1 &= (\tfrac{-1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2})\\
L_2 &= (1,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0)\\
L_3 &= (1,\ -1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0)\\
L_4 &= (1,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0)\\
L_5 &= (1,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0)\\
L_6 &= (1,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0)\\
L_7 &= (1,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0)\\
L_8 &= (1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0)
\end{aligned}$$

The basis $\{M_k\}$. Defined by $M_k := \sigma(L_k \cdot s)$ with $\sigma = (1\ 2)$ acting on coordinates by swapping e_1 and e_2. Spans $\sigma(Ls)$ over $\mathbb{Z}$.

$$\begin{aligned}
M_1 &= (\tfrac{-3}{2},\ \tfrac{-1}{2},\ \tfrac{-1}{2},\ \tfrac{-1}{2},\ \tfrac{-1}{2},\ \tfrac{-1}{2},\ \tfrac{-1}{2},\ \tfrac{-1}{2})\\
M_2 &= (-1,\ 0,\ 0,\ 0,\ 1,\ 0,\ 1,\ 1)\\
M_3 &= (0,\ 1,\ 1,\ 1,\ 0,\ 1,\ 0,\ 0)\\
M_4 &= (-1,\ 0,\ 1,\ 0,\ 0,\ 1,\ 0,\ 1)\\
M_5 &= (-1,\ 1,\ 1,\ 0,\ 0,\ 0,\ 1,\ 0)\\
M_6 &= (-1,\ 1,\ 0,\ 1,\ 0,\ 0,\ 0,\ 1)\\
M_7 &= (-1,\ 0,\ 1,\ 1,\ 1,\ 0,\ 0,\ 0)\\
M_8 &= (-1,\ 1,\ 0,\ 0,\ 1,\ 1,\ 0,\ 0)
\end{aligned}$$

The basis $\{N_k\}$. Defined by $N_k := \sigma(L_k \cdot \bar{s})$. Spans $\sigma(L\bar{s})$ over $\mathbb{Z}$.

$$\begin{aligned}
N_1 &= (2,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0)\\
N_2 &= (0,\ 0,\ -1,\ 0,\ -1,\ 0,\ -1,\ -1)\\
N_3 &= (-1,\ -1,\ 0,\ -1,\ 0,\ -1,\ 0,\ 0)\\
N_4 &= (0,\ -1,\ -1,\ 0,\ 0,\ -1,\ 0,\ -1)\\
N_5 &= (0,\ -1,\ -1,\ -1,\ 0,\ 0,\ -1,\ 0)\\
N_6 &= (0,\ -1,\ 0,\ -1,\ -1,\ 0,\ 0,\ -1)\\
N_7 &= (0,\ 0,\ -1,\ -1,\ -1,\ -1,\ 0,\ 0)\\
N_8 &= (0,\ -1,\ 0,\ 0,\ -1,\ -1,\ -1,\ 0)
\end{aligned}$$

Lemma 4.2: $L \cdot L \subseteq L$.

i	j	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8
1	1	-1	-1	-1	0	0	0	0	0
1	2	0	-2	-2	1	1	0	1	0
1	3	2	2	2	-1	-1	0	-1	0
1	4	0	-2	-2	0	1	1	0	1
1	5	2	1	2	-1	-1	0	0	-1
1	6	0	-1	-2	0	0	0	1	1
1	7	2	1	2	0	-1	-1	-1	0
1	8	2	2	2	-1	0	-1	-1	-1
2	1	2	1	2	-1	-1	0	-1	0
2	2	0	1	-1	0	0	0	0	0
2	3	0	1	1	0	0	0	0	0
2	4	0	0	-1	1	0	1	0	0
2	5	2	3	3	-1	0	-1	-1	-1
2	6	0	1	0	-1	0	1	0	0
2	7	0	0	-1	0	0	0	1	1
2	8	0	1	0	0	0	0	-1	1
3	1	0	-1	-2	1	1	0	1	0
3	2	0	1	1	0	0	0	0	0
3	3	0	-1	1	0	0	0	0	0
3	4	0	0	1	1	0	-1	0	0
3	5	-2	-3	-3	1	2	1	1	1
3	6	0	-1	0	1	0	1	0	0
3	7	0	0	1	0	0	0	1	-1
3	8	0	-1	0	0	0	0	1	1
4	1	2	2	2	-1	-1	-1	0	-1
4	2	0	1	0	1	0	-1	0	0
4	3	0	-1	0	1	0	1	0	0
4	4	0	-1	-1	2	0	0	0	0
4	5	0	-1	-1	1	1	0	1	0
4	6	0	0	-1	1	0	1	0	0
4	7	0	0	0	1	-1	0	1	0
4	8	2	2	3	0	-1	-1	-1	0
5	1	0	-1	-2	1	0	0	0	1
5	2	-2	-2	-4	1	2	1	1	1
5	3	2	2	4	-1	0	-1	-1	-1
5	4	0	0	0	1	1	0	-1	0
5	5	0	-1	-1	0	2	0	0	0
5	6	0	-1	-1	0	1	1	0	1
5	7	0	-1	-1	1	1	0	1	0
5	8	0	0	0	0	1	-1	0	1
6	1	2	1	2	0	0	-1	-1	-1
6	2	0	0	-1	1	0	1	0	0
6	3	0	0	1	-1	0	1	0	0
6	4	0	-1	0	1	0	1	0	0
6	5	0	0	0	0	1	1	0	-1
6	6	0	-1	-1	0	0	2	0	0
6	7	2	2	3	-1	-1	0	0	-1
6	8	0	-1	-1	0	1	1	0	1
7	1	0	-1	-2	0	1	1	0	0
7	2	0	1	0	0	0	0	1	-1
7	3	0	-1	0	0	0	0	1	1
7	4	0	-1	-1	1	1	0	1	0
7	5	0	0	0	-1	1	0	1	0
7	6	-2	-3	-4	1	1	2	2	1
7	7	0	-1	-1	0	0	0	2	0
7	8	0	0	-1	0	0	0	1	1
8	1	0	-2	-2	1	0	1	1	0
8	2	0	0	-1	0	0	0	1	1
8	3	0	0	1	0	0	0	-1	1
8	4	-2	-3	-4	2	1	1	1	2
8	5	0	-1	-1	0	1	1	0	1
8	6	0	0	0	0	-1	1	0	1
8	7	0	-1	0	0	0	0	1	1
8	8	0	-1	-1	0	0	0	0	2

Tab. 1: Coefficients c_k such that $L_i \cdot L_j = \sum_{k=1}^{8} c_k \, L_k$, for the bases defined above.

Lemma 4.3: $L \cdot \sigma(L\bar{s}) \subseteq \sigma(L\bar{s})$.

i	j	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8
1	1	-1	-1	-1	0	0	0	0	0
1	2	0	-2	-2	0	1	1	0	1
1	3	2	2	2	0	-1	-1	0	-1
1	4	0	-1	-2	0	1	0	1	0
1	5	2	1	2	-1	-1	0	0	-1
1	6	0	-2	-2	1	0	0	1	1
1	7	2	2	2	-1	-1	-1	-1	0
1	8	2	1	2	0	0	-1	-1	-1
2	1	2	2	2	-1	-1	-1	0	-1
2	2	-1	0	-2	0	0	1	0	1
2	3	1	1	2	0	0	0	-1	0
2	4	0	0	-1	1	0	0	0	1
2	5	0	0	-1	0	1	1	0	0
2	6	1	1	1	0	-1	1	-1	0
2	7	1	2	2	-1	-1	0	0	0
2	8	1	1	1	-1	0	0	-1	1
3	1	0	-2	-2	1	1	1	0	1
3	2	1	2	2	0	0	-1	0	-1
3	3	-1	-1	0	0	0	0	1	0
3	4	0	0	1	1	0	0	0	-1
3	5	0	0	1	0	1	-1	0	0
3	6	-1	-1	-1	0	1	1	1	0
3	7	-1	-2	-2	1	1	0	2	0
3	8	-1	-1	-1	1	0	0	1	1
4	1	2	1	2	-1	-1	0	-1	0
4	2	0	0	-1	1	0	1	0	0
4	3	0	0	1	1	0	-1	0	0
4	4	0	-1	-1	2	0	0	0	0
4	5	2	2	3	0	0	-1	-1	-1
4	6	0	-1	0	1	0	1	0	0
4	7	0	-1	-1	1	0	0	1	1
4	8	0	0	0	1	0	0	-1	1

i	j	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8
5	1	0	-1	-2	1	0	0	0	1
5	2	0	1	0	0	1	0	-1	0
5	3	0	-1	0	0	1	0	1	0
5	4	-2	-3	-4	2	2	1	1	1
5	5	0	-1	-1	0	2	0	0	0
5	6	0	-1	-1	0	1	1	0	1
5	7	0	0	-1	0	1	0	1	0
5	8	0	0	0	0	1	-1	0	1
6	1	2	1	2	0	0	-1	-1	-1
6	2	1	2	1	-1	0	0	0	-1
6	3	-1	-1	-1	1	0	1	1	0
6	4	0	1	0	1	0	0	0	-1
6	5	0	0	0	0	1	0	1	-1
6	6	-1	-2	-3	1	1	2	1	0
6	7	1	1	1	0	-1	0	1	-1
6	8	1	1	1	0	0	0	0	0
7	1	0	-1	-2	0	1	1	0	0
7	2	-1	-1	-2	0	1	1	1	1
7	3	1	0	2	0	-1	0	0	0
7	4	0	-1	-1	1	0	1	1	0
7	5	0	-1	0	0	1	1	0	0
7	6	-1	-2	-2	0	0	2	1	1
7	7	1	0	1	-1	0	0	1	0
7	8	-1	-3	-3	1	1	1	1	2
8	1	0	-2	-2	1	0	1	1	0
8	2	-1	-1	-3	1	1	0	1	1
8	3	1	2	3	-1	-1	-1	0	0
8	4	0	0	0	1	0	-1	1	0
8	5	0	-1	-1	0	1	0	1	1
8	6	-1	-1	-2	1	0	1	1	1
8	7	-1	-1	-2	0	0	0	2	1
8	8	1	1	1	0	-1	-1	0	1

Tab. 2: Coefficients c_k such that $L_i \cdot N_j = \sum_{k=1}^{8} c_k\, N_k$, for the bases defined above.

Lemma 4.4: $\sigma(Ls)\cdot\sigma(Ls)\subseteq\sigma(Ls)$.

i	j	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8
1	1	-1	1	1	0	0	0	0	0
1	2	0	0	2	0	-1	-1	0	-1
1	3	-2	-2	-4	0	1	1	0	1
1	4	0	1	2	-2	-1	0	-1	0
1	5	-2	-1	-2	1	-1	0	0	1
1	6	0	2	2	-1	0	-2	-1	-1
1	7	-2	-2	-2	1	1	1	-1	0
1	8	-2	-1	-2	0	0	1	1	-1
2	1	-2	-3	-2	0	1	1	0	1
2	2	2	-1	1	0	0	0	0	0
2	3	2	3	3	0	-2	-2	0	-2
2	4	2	1	2	-1	0	-1	0	-2
2	5	4	4	7	-2	-3	-2	-1	-2
2	6	0	-2	-1	1	0	-1	2	0
2	7	2	1	2	0	-1	-2	-1	0
2	8	2	1	3	1	-1	-1	-1	-2
3	1	0	1	0	0	-1	-1	0	-1
3	2	-2	-3	-5	0	2	2	0	2
3	3	2	1	1	0	0	0	0	0
3	4	-2	-3	-6	1	2	1	2	2
3	5	0	-2	-3	0	1	2	1	0
3	6	0	-2	-3	1	0	1	0	2
3	7	2	3	2	-2	-1	0	-1	0
3	8	2	1	1	-1	1	-1	-1	0
4	1	-2	-1	-2	-1	1	0	1	0
4	2	0	-2	-1	-1	0	1	0	2
4	3	4	4	5	-1	-2	-1	-2	-2
4	4	2	1	1	-2	0	0	0	0
4	5	2	2	3	-2	-2	1	-1	-1
4	6	2	1	2	-1	-2	-1	0	0
4	7	4	5	7	-3	-2	-2	-3	-1
4	8	2	0	2	-1	0	0	-1	-1
5	1	0	1	2	-1	-2	0	0	-1
5	2	-2	-5	-6	2	1	2	1	2
5	3	2	3	2	0	-1	-2	-1	0
5	4	0	-1	-2	0	0	-1	1	1
5	5	2	1	1	0	-2	0	0	0
5	6	0	-1	-1	2	-1	-1	0	1
5	7	0	0	-1	0	-1	0	-1	2
5	8	2	2	2	0	-1	-1	-2	-1
6	1	-2	-2	-2	1	0	-1	1	1
6	2	2	1	2	-1	0	-1	-2	0
6	3	2	3	2	-1	0	-1	0	-2
6	4	0	0	-1	-1	2	-1	0	0
6	5	2	2	2	-2	-1	-1	0	-1
6	6	2	1	1	0	0	-2	0	0
6	7	2	4	3	-1	-1	-2	-2	-1
6	8	4	5	7	-2	-1	-3	-2	-3
7	1	0	2	2	-1	-1	-1	-2	0
7	2	0	-2	-1	0	1	2	-1	0
7	3	0	-2	-3	2	1	0	1	0
7	4	-2	-4	-6	1	2	2	1	1
7	5	2	1	2	0	-1	0	-1	-2
7	6	0	-3	-2	1	1	0	0	1
7	7	2	1	1	0	0	0	-2	0
7	8	0	-1	-1	1	2	0	-1	-1
8	1	0	1	2	0	0	-1	-1	-2
8	2	0	-2	-2	-1	1	1	1	0
8	3	0	0	-2	1	-1	1	1	0
8	4	0	1	-1	-1	0	0	1	-1
8	5	0	-1	-1	0	-1	1	2	-1
8	6	-2	-4	-6	2	1	1	2	1
8	7	2	2	2	-1	-2	0	-1	-1
8	8	2	1	1	0	0	0	0	-2

Tab. 3: Coefficients c_k such that $M_i \cdot M_j = \sum_{k=1}^{8} c_k\, M_k$, for the bases defined above.

A.1 The mod-2-quotient reading of Tables 2 and 3

The integer tables above are the explicit certificates of Lemmas 4.3 and 4.4; we now record the structural account of *what* they certify.

From $L\bar{s}\cap Ls = 2L$ (Section 2.3) one has $2L\subseteq L\bar{s}$ and $2L\subseteq Ls$, and applying σ (which fixes $2L$),

$$2L\subseteq\sigma(L\bar{s}),\qquad 2L\subseteq\sigma(Ls).$$

Because $L\cdot L\subseteq L$ (Lemma 4.2) and the octonion product is $\mathbb{Z}$-bilinear, $(a{+}2c)\cdot b\equiv a\cdot b \pmod{2L}$ for $c\in L$, and likewise in the right factor. The product therefore descends to a well-defined $\mathbb{F}_2$-bilinear product $\bar{\mu}$ on the *mod-2 quotient*

$$\bar{L} := L/2L\cong\mathbb{F}_2^8,\qquad \bar{\mu}\colon \bar{L}\times\bar{L}\to\bar{L},$$

the octonion algebra over $\mathbb{F}_2$; the structure constants of $\bar{\mu}$ are exactly the entries of Table 1 reduced modulo 2.

Since $2L$ lies in both $\sigma(L\bar{s})$ and $\sigma(Ls)$, those sublattices are *full preimages*, under $L \to \bar{L}$, of $\mathbb{F}_2$-subspaces

$$V := \sigma(L\bar{s})/2L, \qquad W := \sigma(Ls)/2L,$$

and from $\sigma(L\bar{s}) + \sigma(Ls) = L$ and $\sigma(L\bar{s}) \cap \sigma(Ls) = 2L$ one obtains $\bar{L} = V \oplus W$ with $\dim V = \dim W = 4$. Reducing the closure statements modulo $2L$ then gives

$$\text{Lemma 4.3} \iff \bar{\mu}(\bar{L}, V) \subseteq V, \qquad \text{Lemma 4.4} \iff \bar{\mu}(W, W) \subseteq W.$$

That is: Table 2 certifies that V is a **left ideal** of the octonion $\mathbb{F}_2$-algebra $L/2L$; Table 3 certifies that W is a **subalgebra** of it. The integer tables and this mod-2-quotient description are complementary: the tables verify, the quotient explains. The descent and both equivalences were checked in exact arithmetic (`python_project/src/verify_mod2_quotient.py`).

A note on terminology: the construction is a *quotient*, the surjection $L \to L/2L$, and is named the *mod-2 quotient*. It is kept verbally distinct from a 2-adic scaling or saturation; Corradetti's [Corradetti2026] thematically adjacent 2-primary conductor sublattice of E_8 rescales an order and 2-adically saturates a *sub*lattice rather than passing to a quotient. Both works find 2-primary structure governing octonion-derived orders, but the operations differ.

In addition to Baez/Egan. The lattice-level picture of $\bar{L} \cong \mathbb{F}_2^8$ as an orthogonal space under the plus-type quadratic form induced by the E_8 Gram form – together with its 4-dimensional maximal totally isotropic subspaces (Lagrangians) and the $270 \times 64 = 17{,}280$ factorisation counting Lagrangians and their complementary pairs – is already in Baez/Egan [Baez2014]; see Section 6. Building on that foundation, the present construction equips $\bar{L}$ with the $\mathbb{F}_2$-bilinear product $\bar{\mu}$ inherited from the octonion product $L \cdot L \subseteq L$, and identifies the two Lagrangians of one specific complementary pair – $V = \sigma(L\bar{s})/2L$ and $W = \sigma(Ls)/2L$ – as ideal and subalgebra of $(\bar{L}, \bar{\mu})$.

Polarization of $\bar{L}$ by σ. Direct verification yields two further structural facts about the decomposition $\bar{L} = V \oplus W$. First, V is in fact a *two-sided* ideal of $\bar{L}$: the right-ideal direction $\bar{\mu}(V, \bar{L}) \subseteq V$ holds in addition to the left-ideal direction (Lemma 4.3). The right-ideal direction is not formally implied by the left-ideal direction, since $\bar{\mu}$ is non-commutative in the L-basis.[4]

Second, equip $\bar{L}$ with the natural plus-type quadratic form

$$q\colon \bar{L} \to \mathbb{F}_2, \qquad q(v + 2L) \ := \ N(v)/2 \pmod 2,$$

well-defined because L is even ($N(v + 2u) \equiv N(v) \pmod 4$). For $L = D_8^+$ this is the standard plus-type form on $\mathbb{F}_2^8$ with Witt index 4. Direct verification shows that *both* V and W are totally isotropic under q (every element of either subspace has $q = 0$); each is therefore a maximal totally isotropic subspace – a *Lagrangian.* The decomposition $\bar{L} = V \oplus W$ is thus a *Witt decomposition of* $(\bar{L}, q)$ *into a complementary pair of Lagrangians*, a polarization of the orthogonal $\mathbb{F}_2$-space.

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[4] The standard $\mathbb{F}_2$-octonion-algebra (in the basis $\{e_0, \ldots, e_7\}$) is commutative, because the antisymmetric part of the octonion product satisfies $e_i \cdot e_j - e_j \cdot e_i \in 2\mathbb{O}$, which vanishes modulo 2. In the L-basis, however, $L_1 = s$ is half-integer while $L_2, \ldots, L_8$ are integer, so the antisymmetric part of $L_1 \cdot L_k$ involves half-integer coefficients whose doubling need not lie in $2L$. Hence $L_1 \cdot L_k - L_k \cdot L_1$ need not vanish modulo $2L$, and $\bar{\mu}$ in the L-basis is genuinely non-commutative.

The structural reframing. In this picture, Lemmas 4.3 and 4.4 together express a single fact: *the construction of Section 3 produces a polarization of* $\bar{L}$, with one Lagrangian carrying the two-sided-ideal role (V) and the other carrying the subalgebra role (W). The open question recorded in Section 7 – a structural reason for Lemma 4.4 – accordingly opens into a research program: which polarizations of $\bar{L}$ arise from this construction, and whether any such polarization lifts to a closed bilinear product on Λ itself.

B Historical note: integral Cayley numbers, 1923–1946

The construction this paper builds on was developed across four primary papers between 1923 and 1946. Each was obtained in the original and re-derived in exact arithmetic in this project (`python_project/src/verify_{dickson_1923,kirmse_1924,mahler_1942,coxeter_1946}.py`; verification notes in `paper/reviews/`).

Dickson (1923) [Dickson1923] gave the first construction: the Cayley–Dickson doubled product and an explicit 8-element basis of an order O_D, with all 64 basis products closing. His Theorem XV states three maximal orders; the true count is *seven.* The cause is in his derivation (p. 321): he assumes the orders contain the Hurwitz quaternions, and exactly three of the seven do.

Kirmse (1924) [Kirmse1924] took up the problem independently. His multiplication table is itself a correct octonion algebra, and he derived a correct *necessary* form for the maximal orders – thirty E_8-type lattices realise it – without proving sufficiency. Among the eight of those lattices that are stable under multiplication by units (an unstated narrowing of the criterion, in our reconstruction), he exhibited the one, J_1 (p. 70), that is not an order. The remaining seven of his eight are the genuine maximal orders. Kirmse also develops an ideal-theoretic treatment of the integral octonions – a substantive companion contribution that is taken up again in Mahler (1942) – and methodically takes algebra alternativity as a foundational constraint, anticipating the structural role alternativity plays in later integral-octonion arithmetic.

Mahler (1942) [Mahler1942] developed the arithmetic of a maximal order: for every Cayley number X there is an integral G with $N(X-G) \leq \frac{1}{2}$ (sharper than Dickson's 5/4), and every left or right ideal is principal, generated by an integral Cayley number of norm 1, 2, or 4.

Coxeter (1946) [Coxeter1946] identifies Kirmse's error with the same non-closure witness and records that the remedy is due to R. H. Bruck: "Bruck's domain J can be derived from J_1 by transposing two of the i's" (§4). This is a transposition of two imaginary basis units, a linear involution of $\mathbb{R}^8$; the $\binom{7}{2} = 21$ such transpositions yield the seven maximal orders in seven sets of three. His §§6–13 develop the $E_8/5_{21}$ geometry of the corrected order. A postscript (§14) records his after-the-fact awareness of Dickson and Mahler and identifies Dickson's undercount with exactly the cause above.

It is curious to find that the Coxeter–Bruck transposition would come in useful here: the same linear involution of $\mathbb{R}^8$, acting now on a different object. The Coxeter–Bruck transposition acts on a maximal-order candidate that sits asymmetrically in $\mathbb{R}^8$; the present σ acts on the coordinate-symmetric placement $L = D_8^+$ (where it is inert at the E_8 stratum) and does its work on the sublattice Ls of Wilson's Leech construction.

C Research methodology

This research was conducted as a collaboration between a human researcher and an AI agent (Claude, Anthropic) over a series of directed prompts, governed by a written methodology (the

"Manifesto") established at the outset.

The human researcher (the author) brought the initial hypothesis – motivated by prior work on Okubo algebras and autotopies of normed composition algebras – established the research methodology, and maintained control of the inquiry through prompt-by-prompt review. The AI agent performed the computational work: implementing octonion algebras, Wilson's Leech lattice construction, trial algebras, and closure tests, and contributed mathematical analysis.

Tests were written from first principles to independently verify every foundation: that octonion multiplication satisfies the composition property $N(xy) = N(x)N(y)$, that Wilson's construction produces a lattice with the correct minimal shell of 196,560 vectors, and that the E_8 lattice is indeed a maximal order. This independent verification is how we know that the properties stated in the literature are correctly implemented. Closure of the twisted product was first observed on over 12,000,000 random pairs of minimal vectors with zero failures, which prompted the search for a symbolic proof.

Symbolic proof and numeric validation. The proof of Theorem 1.1 given in Section 4 is symbolic: each of the four lemmas is established by exhibiting a finite list of integer coefficients (Appendix A), and the conclusion follows by $\mathbb{Z}$-bilinearity. No floating-point or sampled arithmetic enters the proof. Independently, the same basis-by-basis identities that constitute the appendix tables were recomputed in two separate computer-algebra implementations as a cross-check: a Python implementation using `fractions.Fraction` exact rationals (`python_project/src/symbolic_proof_checks.py`), and a GAP/LOOPS implementation using GAP's native exact rational arithmetic (`gap_project/tests/test_lemmas.g`). Both implementations reproduce the same integer coefficients displayed in the appendix. This redundancy is validation, not proof: it serves to detect transcription or implementation errors in the appendix tables, but the logical content of the proof rests on the tables themselves.

The methodology required:

- Every claim backed by a reference or computation.
- Every prompt logged, with actions recorded.
- Every conclusion independently verified.
- Forward-evolving corrections: no rewriting history; errors are corrected in subsequent entries.

The path to the result was not direct. Six trial algebra families were explored and ruled out before the transposition twist was identified in trial 007. The systematic methodology – understanding the anatomy of failure (which conditions fail, for which vector types, and why) before trying the next approach – was essential. The failure analysis of the untwisted product (Wilson's condition (3) per Definition 2.1, exclusively type-3×type-3) directly pointed to the mechanism that the twist needed to correct.

Iterative convergence toward a robust result – including learning along the way what it is we are actually doing – is how study, research, and discovery have always proceeded. Mistakes were made on both sides: the AI agent occasionally departed from the agreed protocol or produced incorrect intermediate analyses, and the human author at times framed prompts under misconceptions or accepted intermediate results without sufficient scrutiny. In each case, the error was caught by the systematic cross-checking that the methodology requires – explicit verification, prompt-by-prompt review, and independent reimplementation in two computer-algebra systems – and corrected forward. What is novel here is the ability to trace the entire path, beginning to end, through the complete list of prompts.

The full record – all prompts, code, evidence, and git history – is available at the repository [repo]. The reader can form their own assessment of what this means for questions of discovery and attribution.

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