

An order on the Leech lattice from a \mathbb{Z}_3 -symmetric triple-octonion product

29 April 2026 (v3)

Jens Köpflinger

105 E Avondale Dr, Greensboro, NC 27403, USA
jenskoeplinger@gmail.com

Abstract

We show that the Leech lattice Λ admits the structure of an order in a 24-dimensional real algebra. The product is a \mathbb{Z}_3 -symmetric triple-octonion product on \mathbb{R}^{24} , assembled from nine octonion-multiplication blocks under cyclic \mathbb{Z}_3 permutation of three octonion factors. Realising this product concretely on Wilson’s sublattice description of $\Lambda \subset \mathbb{R}^{24}$ requires a particular Coxeter representation of the octonions, related to the standard Fano-plane labelling by a single transposition σ of two imaginary basis indices. The chain Kirmse \rightarrow Coxeter $\rightarrow \sigma$ — each step a transposition of two basis indices — successively aligns the octonion multiplication with the lattice: Coxeter’s transposition turns Kirmse’s E_8 candidate into an order in the octonions, and σ aligns the triple-octonion product so that it closes on Λ .

This result was developed through a systematic human–AI collaboration; the complete research record is publicly available [repo].

1 Introduction

The Leech lattice Λ is the unique even unimodular lattice of rank 24 with no vectors of squared norm 2 [ConwaySloane1999]. Its minimal shell $\text{Min}(\Lambda)$ consists of 196,560 vectors of squared norm 8, and its automorphism group is the Conway group Co_0 , a double cover of the sporadic simple group Co_1 .

In rank 8, the E_8 lattice L carries an octonionic multiplication that makes it a maximal order in the real octonion division algebra. Seven octonion-multiplication conventions on \mathbb{R}^8 have this property; the resulting structures are in bijection with the seven maximal orders of the integral octonions [Coxeter1946, ConwaySmith2003]. Wilson [Wilson2009, Section 3] showed that Λ embeds naturally in L^3 via three sublattice conditions, raising the question: does Λ inherit an algebraic structure from the octonion product?

This question has been studied from several angles. Dixon [Dixon2010] constructed Λ using an octonionic XY-product. Baez and Egan [Baez2014] explored Jordan-algebraic structures on Λ within the exceptional Jordan algebra $\mathfrak{h}_3(\mathbb{O})$, finding closure under a doubled Jordan product.

The \mathbb{Z}_3 -symmetric triple-octonion product on L^3 — using the same octonion multiplication in all three blocks with \mathbb{Z}_3 -symmetric cross-block routing — satisfies two of Wilson’s three sublattice

conditions automatically, but closure on the third requires a block-sum sublattice Ls to be preserved under octonion multiplication, and that condition is sensitive to which Coxeter representation of the octonions is used in \mathbb{R}^{24} .

The main result of this paper is that the \mathbb{Z}_3 -symmetric triple-octonion product closes on Λ once the octonions are realised in the Coxeter representation that is aligned with Wilson’s sublattice description of Λ . Relative to the standard Fano-plane labelling we use throughout for the E_8 block, that representation is obtained by a single transposition σ of two imaginary basis indices: the resulting product maps the block-sum sublattice Ls into $\sigma(Ls)$, a different sublattice of L that is closed under the underlying octonion multiplication.

Theorem 1.1. *Let \cdot_σ denote the transposition-twisted octonion multiplication on \mathbb{R}^8 (Definition 3.1), and let \star denote the associated triple product on $\mathbb{R}^{24} = \mathbb{O}_1 \oplus \mathbb{O}_2 \oplus \mathbb{O}_3$ with \mathbb{Z}_3 cross-block routing (Definition 3.6). Then for all $u, v \in \Lambda$, the product $u \star v$ lies in Λ .*

Corollary 1.2. $(\Lambda, +, \star)$ is an order in the \mathbb{R} -algebra $(\mathbb{R}^{24}, +, \star)$.

Proof. The product \star is bilinear (it is a sum of bilinear octonion products). By Theorem 1.1, \star maps $\Lambda \times \Lambda$ into Λ . \square

Remark 1.3 (Order-closure, not shell-closure). The closure statement $\Lambda \star \Lambda \subseteq \Lambda$ means that products land somewhere in Λ ; it does not assert that the minimal shell is preserved. Empirically, the product norm on $\text{Min}(\Lambda) \times \text{Min}(\Lambda)$ takes values in $\{16, 32, 48, 64, 80, 96, 112, 128\}$ (see Section 5), none of which equals the minimal value $N = 8$: products of minimal vectors always land on strictly higher shells. This is consistent with $(\Lambda, +, \star)$ being an order in the ring-theoretic sense — a subring that is a free \mathbb{Z} -module of rank equal to the ambient algebra’s dimension — a property weaker than, and independent of, any shell-level closure.

2 Preliminaries

2.1 The octonion algebra

Let \mathbb{O} denote the real octonion algebra with basis $\{e_0, e_1, \dots, e_7\}$, where e_0 is the identity. The multiplication is determined by the seven *Fano triples*

$$(1, 2, 4), (2, 3, 5), (3, 4, 6), (4, 5, 7), (5, 6, 1), (6, 7, 2), (7, 1, 3), \quad (2.1)$$

encoding the rule: for each triple (a, b, c) , $e_a \cdot e_b = +e_c$, with cyclic permutations carrying a positive sign and anti-cyclic permutations carrying a negative sign [Baez2002]. Every imaginary unit squares to $-e_0$. The algebra \mathbb{O} is the unique real normed division algebra of dimension 8 [Baez2002].

The *norm form* is $N(x) = x\bar{x} = \sum_{i=0}^7 x_i^2$ for $x = \sum_i x_i e_i$, where $\bar{x} = x_0 e_0 - \sum_{i=1}^7 x_i e_i$ is the conjugate. The composition property $N(xy) = N(x)N(y)$ holds [Baez2002].

2.2 The E_8 lattice

In Wilson’s setup [Wilson2009, Section 2], the E_8 lattice is $L = D_8^+$, the set of vectors in \mathbb{R}^8 whose coordinates (in the basis $\{e_0, \dots, e_7\}$) are either all integers with even coordinate sum, or all half-integers with odd coordinate sum [ConwaySmith2003]. It has 240 roots (vectors of squared norm 2). Under the standard octonion multiplication, L is a maximal order in \mathbb{O} : it is closed under multiplication [Coxeter1946, ConwaySmith2003].

Remark 2.1 (Kirmse's convention versus ours). The closure $L \cdot L \subseteq L$ depends on which Fano-plane labelling defines the octonion product. Kirmse [Kirmse1924] originally proposed a specific labelling under which D_8^+ is *not* closed — a subtle error corrected by Coxeter [Coxeter1946], who showed that permuting two basis indices yields a different multiplication on the same \mathbb{R}^8 under which D_8^+ is closed. There are seven such Coxeter-corrected conventions, each giving one of the seven maximal orders of the integral octonions [ConwaySmith2003]; see also Petersson [Petersson2018] for a modern account.

The “standard” Fano triples (2.1) used in this paper are one of the Coxeter-corrected conventions, not Kirmse's. We have verified $L \cdot L \subseteq L$ directly under our convention: using an explicit \mathbb{Z} -basis of L and exact rational arithmetic, all 64 basis products are integer combinations of the basis.

Remark 2.2 (Closure of the lattice, not the shells). When we say L is closed under multiplication, we mean closure of the lattice L , not of any individual shell. Since the octonion product is norm-multiplicative, the product of two roots (squared norm 2) has squared norm 4 and lies on the second shell of L , not among the roots.

The same distinction will apply to $(\Lambda, +, \star)$ in Theorem 1.1: products of minimal Leech vectors land on higher shells of Λ , not back in the minimal shell (see Section 5).

2.3 Wilson's Leech lattice

Define $s = \frac{1}{2}(-e_0 + e_1 + \dots + e_7)$ and $\bar{s} = \frac{1}{2}(-e_0 - e_1 - \dots - e_7)$, both of norm $N(s) = N(\bar{s}) = 2$. Then $s \in L$ (it is a type-2 root, with an odd coordinate sum), while $\bar{s} \notin L$ (its coordinate sum is -4 , which falls in the excluded even branch of the half-integer case). Let

$$L\bar{s} = \{\ell \cdot \bar{s} : \ell \in L\}, \quad Ls = \{\ell \cdot s : \ell \in L\}.$$

Both $L\bar{s}$ and Ls are sublattices of L ,¹ with $[L : L\bar{s}] = [L : Ls] = 16$, $L\bar{s} + Ls = L$, and $L\bar{s} \cap Ls = 2L$ [Wilson2009, Section 2].

Definition 2.3 (Wilson [Wilson2009, Section 3]). The *Leech lattice* is

$$\Lambda = \{(x, y, z) \in L^3 : (1) x, y, z \in L; (2) x+y, x+z, y+z \in L\bar{s}; (3) x+y+z \in Ls\}.$$

The 196,560 minimal vectors of Λ decompose into three families [Wilson2009, Section 3]:

Family	Formula	Count
Type 1	$(2\lambda, 0, 0)$ and cyclic permutations	$3 \times 240 = 720$
Type 2	$(\lambda\bar{s}, (\lambda\bar{s})j, 0)$ and cyclic perms	$3 \times 240 \times 16 = 11,520$
Type 3	$((\lambda s)j, \lambda k, (\lambda j)k)$ and cyclic perms	$3 \times 240 \times 16^2 = 184,320$

Here λ ranges over the 240 roots of L , and $j, k \in J = \{\pm e_t : 0 \leq t \leq 7\}$ (16 unit octonions).

3 The construction

Definition 3.1 (Transposition-twisted octonion algebra). Let $\sigma = (i j)$ be a transposition on $\{1, \dots, 7\}$. Extend σ to a linear map $\sigma : \mathbb{R}^8 \rightarrow \mathbb{R}^8$ that fixes e_0 and permutes $e_i \leftrightarrow e_j$. Apply σ

¹ $L\bar{s} \subseteq L$ even though $\bar{s} \notin L$: each $\ell \cdot \bar{s}$ with ℓ a root of L lies in L by direct computation, and the inclusion extends by \mathbb{Z} -linearity in the left factor. $L\bar{s}$ is therefore a \mathbb{Z} -subgroup of L , and is moreover closed under left-multiplication by L (Wilson's condition (2) for the untwisted product, implicit in the sublattice argument of [Wilson2009, Section 3]). For Ls both properties are immediate from $s \in L$.

to each standard Fano triple: $(a, b, c) \mapsto (\sigma(a), \sigma(b), \sigma(c))$. The resulting triples define a new multiplication table on \mathbb{R}^8 . Equivalently, the σ -twisted multiplication is

$$x \cdot_{\sigma} y = \sigma(\sigma(x) \cdot \sigma(y)), \quad (3.1)$$

where \cdot denotes the standard octonion product.

Proposition 3.2. $(\mathbb{R}^8, \cdot_{\sigma})$ is an octonion algebra, isomorphic to (\mathbb{O}, \cdot) via the map σ itself:

$$\sigma(x \cdot y) = \sigma(x) \cdot_{\sigma} \sigma(y) \quad \text{for all } x, y \in \mathbb{R}^8. \quad (3.2)$$

In particular, \cdot_{σ} satisfies the composition property $N(x \cdot_{\sigma} y) = N(x)N(y)$.

Proof. For any $x, y \in \mathbb{R}^8$:

$$\sigma(x) \cdot_{\sigma} \sigma(y) = \sigma(\sigma(\sigma(x)) \cdot \sigma(\sigma(y))) = \sigma(x \cdot y),$$

since $\sigma^2 = \text{id}$. Thus $\sigma : (\mathbb{O}, \cdot) \rightarrow (\mathbb{R}^8, \cdot_{\sigma})$ is a vector-space isomorphism that preserves multiplication. Composition follows. \square

Remark 3.3 (σ vs. a Fano-line permutation). The map σ is a linear involution of \mathbb{R}^8 , swapping two coordinate axes and preserving both the underlying vector space and the Euclidean norm. This is structurally different from a permutation of Fano triples (such as the one Coxeter used to correct Kirmse's E_8 convention), which rewrites the multiplication table without being induced by a linear map on \mathbb{R}^8 . Most coordinate transpositions do not correspond to Fano-triple permutations: $\sigma = (1\ 2)$ does not send the triple $(2, 3, 5)$ in (2.1) to any of the other six listed there, so the Fano line structure is not preserved by σ . Coxeter's correction and the present σ thus act at different strata of the construction; we expand on this in Section 6.

To make the action of σ on the multiplication concrete, we record the count of changed entries.

Remark 3.4. For the transposition $\sigma = (1\ 2)$, exactly 30 of the 64 multiplication table entries change relative to the standard table: 6 entries undergo a sign change only, 16 undergo a target-index change only, and 8 undergo both.

Remark 3.5. The 21 transpositions of $\{1, \dots, 7\}$ form a single orbit under the Fano-plane automorphism group $\text{GL}(3, \mathbb{F}_2)$ (which has order 168 and acts on the 7 points of the Fano plane). Indeed, $\text{GL}(3, \mathbb{F}_2)$ is 2-transitive on the 7 points, so any pair of imaginary indices can be mapped to any other pair, and conjugation of $\sigma = (i\ j)$ by a Fano automorphism produces another transposition. All 21 transpositions are therefore equivalent up to a relabelling of the imaginary basis, and the construction is essentially unique *among transposition twists*; longer-cycle coordinate permutations are not covered by this argument and may yield inequivalent products.

Definition 3.6 (Triple product with \mathbb{Z}_3 routing). Decompose $\mathbb{R}^{24} = \mathbb{O}_1 \oplus \mathbb{O}_2 \oplus \mathbb{O}_3$ into three copies of \mathbb{R}^8 (blocks indexed $\alpha = 1, 2, 3$). For $u = (x, y, z)$ and $v = (x', y', z')$ with $x, x' \in \mathbb{O}_1$, etc., define the *triple product*

$$u \star v = (P_1, P_2, P_3), \quad (3.3)$$

where each block P_{α} is the sum of three σ -twisted octonion products:

$$P_1 = x \cdot_{\sigma} x' + z \cdot_{\sigma} y' + y \cdot_{\sigma} z', \quad (3.4)$$

$$P_2 = y \cdot_{\sigma} y' + x \cdot_{\sigma} z' + z \cdot_{\sigma} x', \quad (3.5)$$

$$P_3 = z \cdot_{\sigma} z' + y \cdot_{\sigma} x' + x \cdot_{\sigma} y'. \quad (3.6)$$

The routing rule is: a product of blocks α and β lands in block γ , where $\{\alpha, \beta, \gamma\} = \{1, 2, 3\}$ for $\alpha \neq \beta$, and in block α when $\alpha = \beta$. All nine block-pair products use the same multiplication \cdot_{σ} .

4 Proof of closure

We prove Theorem 1.1 by showing that the twisted product preserves each of Wilson's three conditions (Definition 2.3). The proof rests on four lemmas about the interaction between the transposition σ , the maximal order L , and the sublattices Ls and $L\bar{s}$.

4.1 Reduction to standard products

Since each σ -twisted product has the form $a \cdot_{\sigma} b = \sigma(\sigma(a) \cdot \sigma(b))$, and σ is linear, each block P_{α} is σ applied to a sum of standard octonion products of σ -images of the inputs.

Write $X = \sigma(x)$, $Y = \sigma(y)$, $Z = \sigma(z)$, $X' = \sigma(x')$, $Y' = \sigma(y')$, $Z' = \sigma(z')$. Then:

$$P_1 = \sigma(X \cdot X' + Z \cdot Y' + Y \cdot Z'), \quad (4.1)$$

$$P_2 = \sigma(Y \cdot Y' + X \cdot Z' + Z \cdot X'), \quad (4.2)$$

$$P_3 = \sigma(Z \cdot Z' + Y \cdot X' + X \cdot Y'). \quad (4.3)$$

Taking sums:

- **Block sum.** $P_1 + P_2 + P_3 = \sigma((X + Y + Z) \cdot (X' + Y' + Z'))$.
- **Pairwise sums.** For example, $P_1 + P_2 = \sigma(X \cdot (X' + Z') + Y \cdot (Y' + Z') + Z \cdot (X' + Y'))$, and similarly for the other two pairs.

4.2 The four lemmas

Lemma 4.1. $\sigma(L) = L$.

Proof. The lattice $L = D_8^+$ consists of all vectors in \mathbb{R}^8 whose coordinates are either all integers with even coordinate sum, or all half-integers with odd coordinate sum (Section 2). The transposition σ permutes two coordinates, which preserves both the integer/half-integer parity and the coordinate sum. Hence $\sigma(L) \subseteq L$. Since σ is an involution, $\sigma(L) = L$. \square

For each of the next three lemmas, L , $\sigma(L\bar{s})$, and $\sigma(Ls)$ are full-rank \mathbb{Z} -lattices of rank 8 in \mathbb{R}^8 . By \mathbb{Z} -bilinearity of the octonion product, an inclusion $A \cdot B \subseteq C$ between such lattices holds if and only if it holds on a chosen pair of \mathbb{Z} -bases: if $\{a_i\}_{i=1}^8$ generates A and $\{b_j\}_{j=1}^8$ generates B , then $A \cdot B \subseteq C$ is equivalent to each of the 64 products $a_i \cdot b_j$ lying in $\mathbb{Z}\langle c_1, \dots, c_8 \rangle$ for a chosen \mathbb{Z} -basis $\{c_k\}_{k=1}^8$ of C . (The forward direction is the substantive one and uses bilinearity; the reverse direction is immediate since each $a_i \in A$ and each $b_j \in B$.) Each lemma below is proved by exhibiting that finite list of integer coefficients explicitly. Concrete \mathbb{Z} -bases for L , $\sigma(Ls)$, and $\sigma(L\bar{s})$ are fixed in Appendix A (denoted there L_k , M_k , N_k respectively); the 64-entry coefficient tables for the three lemmas appear as Tables 1, 2, and 3.

Lemma 4.2. $L \cdot L \subseteq L$.

Proof. The 64 products $L_i \cdot L_j$ are integer combinations of $\{L_1, \dots, L_8\}$; the explicit coefficients are Table 1. By \mathbb{Z} -bilinearity, $L \cdot L \subseteq L$. This recovers the classical fact that $L = D_8^+$ is a maximal order in \mathbb{O} [Coxeter1946, ConwaySmith2003]. \square

Lemma 4.3. $L \cdot \sigma(L\bar{s}) \subseteq \sigma(L\bar{s})$.

Proof. The 64 products $L_i \cdot N_j$ are integer combinations of $\{N_1, \dots, N_8\}$; the explicit coefficients are Table 2. By \mathbb{Z} -bilinearity, $L \cdot \sigma(L\bar{s}) \subseteq \sigma(L\bar{s})$. \square

Lemma 4.4. $\sigma(Ls) \cdot \sigma(Ls) \subseteq \sigma(Ls)$.

Proof. The 64 products $M_i \cdot M_j$ are integer combinations of $\{M_1, \dots, M_8\}$; the explicit coefficients are Table 3. By \mathbb{Z} -bilinearity, $\sigma(Ls) \cdot \sigma(Ls) \subseteq \sigma(Ls)$. \square

Remark 4.5. The sublattice Ls is *not* closed under the standard octonion product: there exist basis vectors $a, b \in Ls$ with $a \cdot b \notin Ls$. This is precisely the obstruction that the twist resolves: it moves the block-sum sublattice from Ls to $\sigma(Ls)$, where closure holds by Lemma 4.4.

Note that $\sigma(Ls)$ is genuinely different from Ls (not just a relabelling). For example, with $\sigma = (1\ 2)$, the element $v = (-1, 0, 1, 0, 0, 1, 0, 1)$ lies in $\sigma(Ls)$ but direct computation shows it is not an integer combination of any \mathbb{Z} -basis for Ls , so $v \notin Ls$. This distinction is what makes the twist useful: σ is not the identity on the block-sum sublattice, so it can redirect a product into a different (and, as it turns out, closed) sublattice.

Remark 4.6 (Two equivalent framings). Let \star_0 denote the analogous triple product built from the *standard* (untwisted) octonion multiplication, i.e. with \cdot replacing \cdot_σ in every block-pair product of Definition 3.6. Applying σ block-wise gives an algebra isomorphism $\Sigma := \sigma \oplus \sigma \oplus \sigma : (\mathbb{R}^{24}, \star) \rightarrow (\mathbb{R}^{24}, \star_0)$, since each block P_α of $u \star v$ is by construction (4.1)–(4.3) the image under σ of the corresponding block of $\Sigma(u) \star_0 \Sigma(v)$. Consequently the central Theorem 1.1 has two equivalent readings:

$$\Lambda \star \Lambda \subseteq \Lambda \iff \Sigma(\Lambda) \star_0 \Sigma(\Lambda) \subseteq \Sigma(\Lambda).$$

Neither reading asserts $\Lambda \star_0 \Lambda \subseteq \Lambda$, which fails: the standard triple product obstructs Wilson's condition (3), since it would require $Ls \cdot Ls \subseteq Ls$. The twist does not rename a closure that was already present; it identifies a distinct Leech embedding $\Sigma(\Lambda) \neq \Lambda$ on which the standard triple product closes. To see $\Sigma(\Lambda) \neq \Lambda$ explicitly: an element $(x', y', z') \in \Sigma(\Lambda)$ is by construction $(\sigma(x), \sigma(y), \sigma(z))$ for some $(x, y, z) \in \Lambda$, so its block sum $x' + y' + z' = \sigma(x + y + z)$ lies in $\sigma(Ls)$. Since the block-sum map $\Lambda \rightarrow Ls$ is surjective (Wilson [Wilson2009, Section 3]) and $\sigma(Ls) \neq Ls$ (Remark 4.5), there exist $(x, y, z) \in \Lambda$ with $\sigma(x + y + z) \in \sigma(Ls) \setminus Ls$, and the corresponding triple in $\Sigma(\Lambda)$ violates Wilson's condition (3) for membership in Λ .

4.3 Condition (1): each block lies in L

Proposition 4.7. *For all $u, v \in \Lambda$, each block P_α of $u \star v$ lies in L .*

Proof. Consider $P_1 = \sigma(X \cdot X' + Z \cdot Y' + Y \cdot Z')$. Since $u \in \Lambda$, condition (1) gives $x \in L$, so $X = \sigma(x) \in L$ by Lemma 4.1. Similarly $X', Y, Y', Z, Z' \in L$. By Lemma 4.2, each product $X \cdot X'$, $Z \cdot Y'$, $Y \cdot Z'$ lies in L . Their sum lies in L (closed under addition). Applying σ and using Lemma 4.1, $P_1 \in L$. The same argument applies to P_2 and P_3 . \square

4.4 Condition (2): pairwise sums lie in $L\bar{s}$

Proposition 4.8. *For all $u, v \in \Lambda$, the pairwise sums $P_\alpha + P_\beta$ lie in $L\bar{s}$.*

Proof. We show $P_1 + P_2 \in L\bar{s}$; the other two cases follow by the \mathbb{Z}_3 symmetry of the routing.

From (4.1)–(4.2):

$$P_1 + P_2 = \sigma(X \cdot (X' + Z') + Y \cdot (Y' + Z') + Z \cdot (X' + Y')).$$

Since $v \in \Lambda$, condition (2) gives $x' + z' \in L\bar{s}$, so $X' + Z' = \sigma(x' + z') \in \sigma(L\bar{s})$. Similarly $Y' + Z' \in \sigma(L\bar{s})$ and $X' + Y' \in \sigma(L\bar{s})$. Since $u \in \Lambda$, we have $X, Y, Z \in L$ (by Lemma 4.1).

By Lemma 4.3, each product $X \cdot (X' + Z')$, $Y \cdot (Y' + Z')$, $Z \cdot (X' + Y')$ lies in $\sigma(L\bar{s})$. Their sum lies in $\sigma(L\bar{s})$. Applying the outer σ :

$$P_1 + P_2 = \sigma(\text{element of } \sigma(L\bar{s})) \in \sigma(\sigma(L\bar{s})) = L\bar{s},$$

since $\sigma^2 = \text{id}$. □

4.5 Condition (3): the block sum lies in Ls

This is the crux of the proof. For the standard (untwisted) product the block sum would require $Ls \cdot Ls \subseteq Ls$, which does not hold (Remark 4.5). The twist redirects the computation through $\sigma(Ls)$, where closure holds.

Proposition 4.9. *For all $u, v \in \Lambda$, the block sum $P_1 + P_2 + P_3$ lies in Ls .*

Proof. From (4.1)–(4.3):

$$P_1 + P_2 + P_3 = \sigma((X + Y + Z) \cdot (X' + Y' + Z')).$$

Since $u \in \Lambda$, condition (3) gives $x + y + z \in Ls$, so

$$X + Y + Z = \sigma(x + y + z) \in \sigma(Ls).$$

Similarly $X' + Y' + Z' \in \sigma(Ls)$.

By Lemma 4.4, $\sigma(Ls) \cdot \sigma(Ls) \subseteq \sigma(Ls)$, so

$$(X + Y + Z) \cdot (X' + Y' + Z') \in \sigma(Ls).$$

Applying the outer σ :

$$P_1 + P_2 + P_3 \in \sigma(\sigma(Ls)) = Ls. \quad \square$$

This completes the proof of Theorem 1.1.

5 Algebraic properties

The order $(\Lambda, +, \star)$ has been tested for several standard algebraic properties. All percentages below are over samples drawn uniformly from $\text{Min}(\Lambda) \times \text{Min}(\Lambda)$ (or from $\text{Min}(\Lambda)$, for power-associativity); sample sizes are 5,000 pairs for norm multiplicativity and 2,000 pairs (resp. 2,000 elements for power-associativity) for each remaining test.

Property	Holds?	Details
Multiplicative identity	No	the ambient algebra $(\mathbb{R}^{24}, +, \star)$ has no identity element
Commutativity	No	$< 0.1\%$ of pairs commute
Norm multiplicativity	No	$\approx 47\%$ of pairs satisfy $N(u \star v) = N(u)N(v)$
Left alternativity	No	$< 0.1\%$ of pairs satisfy $(x \star x) \star y = x \star (x \star y)$
Right alternativity	No	$< 0.1\%$ of pairs satisfy $(x \star y) \star y = x \star (y \star y)$
Flexibility	No	$\approx 8.6\%$ of pairs satisfy $(x \star y) \star x = x \star (y \star x)$
Power-associativity	No	$\approx 29\%$ of elements satisfy $x^2 \star x = x \star x^2$

Remark 5.1. The product norm on $\text{Min}(\Lambda) \times \text{Min}(\Lambda)$ takes values in $\{16, 32, 48, 64, 80, 96, 112, 128\}$, all multiples of 16. The mode is 64 (the norm-multiplicative value $N(u)N(v) = 8 \times 8$).

Remark 5.2. The ambient algebra $(\mathbb{R}^{24}, +, \star)$ has no multiplicative identity: any putative left identity (a, b, c) would have to act on an arbitrary (x', y', z') by the block matrix $\begin{pmatrix} a & c & b \\ c & b & a \\ b & a & c \end{pmatrix}$ (entries acting by left-multiplication in \mathbb{O}), and no triple (a, b, c) makes this the identity on all three blocks simultaneously. In particular, (e_0, e_0, e_0) is not an identity: it maps (x', y', z') to $(x' + y' + z', x' + y' + z', x' + y' + z')$. Consequently $(\Lambda, +, \star)$ is a non-unital order.

The order does not satisfy any of the classical algebraic identities (associativity, alternativity, flexibility, power-associativity, commutativity). The defining property is *closure*: the Leech lattice is stable under the product.

6 Related work

Wilson (2009) [Wilson2009] characterises Λ as $\{(x, y, z) \in L^3 : \text{conditions 1–3}\}$, gives an explicit listing of the 196,560 minimal vectors in the form $3 \times 240 \times (1 + 16 + 16 \times 16)$, and proves that Λ is even and self-dual. The sublattice framework is the setting used in the present paper.

Dixon (2010) [Dixon2010] constructs Λ using the octonion XY-product²

$$A \circ_{XY} B = (AX)(Y^\dagger B),$$

where $X, Y \in \mathbb{O}$ are chosen so that YX^\dagger is a unit octonion (the identity of the new multiplication). When $Y^\dagger = X^{-1}$ the XY-product specialises to the X-product $A \circ_X B = (AX)(X^{-1}B)$; Dixon uses this X-product with $X = \ell_0 = \frac{1}{2}(1 + e_1 + \dots + e_7)$ to obtain E_8 , and general XY-products for higher-dimensional laminated lattices including Λ . We have independently reproduced this construction in the supplemental code [repo].

Baez and Egan (2014) [Baez2014] study closure properties of integer sublattices of the exceptional Jordan algebra $\mathfrak{h}_3(\mathbb{O})$. In parts 9–10 of the series, Baez considers the 27-dimensional lattice $\mathbb{Z}^3 \oplus L_L \subset \mathfrak{h}_3(\mathbb{O})$, with integers on the diagonal and an off-diagonal Leech sublattice $L_L \cong \Lambda$ embedded as an octonionic triple. This lattice is *not* closed under the Jordan product $x \circ y = xy + yx$, but *is* closed under the doubled product $2(xy + yx)$. Egan further enumerated at least 17,280 distinct off-diagonal Leech embeddings $L_L \subset \mathbb{O}^3$ for which the associated 27-dimensional lattice

² The construction has an earlier precursor in Dixon [Dixon1995], where a closely related octonionic description of Λ is proposed. The formula there required a small correction in the third family of minimal vectors, which Wilson [Wilson2009, end of §5] supplied in a footnote-style remark while independently deriving the sublattice characterisation. The situation is similar to the Kirmse–Coxeter relationship discussed below: an early, structurally correct proposal needing correction.

is closed under this doubled product (see part 10 of [Baez2014] for Egan’s computational counts). Baez left open the question whether any such lattice is closed under the undoubled Jordan product — which would remove the factor-of-2 discrepancy — and conjectures that the answer is negative. We have independently reproduced the full Baez–Egan verification chain — the E_8 simple-roots data, the lemma $L_1 \cap L_2 = 2L_0$, the Leech-lattice theorem, the doubled-Jordan-product closure, the derivation of the lower bound 244,035,421 on the number of E_8^3 -orbit Leech embeddings, and Egan’s sharper count of 17,280 Jordan rings. This last factors as $17,280 = 270 \times 64$: the 270 is the number of sublattices $L_1 \subset L$ isometric to $\sqrt{2}L$, equivalently the number of 4-dimensional maximal totally isotropic subspaces of $L/2L \cong \mathbb{F}_2^8$ under the plus-type quadratic form induced by the E_8 Gram form; the 64 is the number of L_2 in this collection that meet a given L_1 only in $2L$, equivalently the number of 4-dimensional complementary isotropic subspaces. All Baez–Egan claims hold under our reading. Details and code are in [repo].

Comparison. The product \star is bilinear on \mathbb{R}^{24} itself. It does not require an enclosing 27-dimensional Jordan-algebraic ambient within which the Leech sits as a subspace (cf. [Baez2014]), and the cross-block \mathbb{Z}_3 symmetry is supplied by the routing of Definition 3.6 rather than by the octonion product itself: Dixon’s XY-product [Dixon1995, Dixon2010] is a useful device for describing each E_8 block, but it is not in itself \mathbb{Z}_3 -symmetric across the three factors. Whether $\Sigma(\Lambda)$ overlaps with one of the embeddings catalogued by Egan, or sits outside that family, is a question this paper does not address.

Anatomy of the Baez–Egan closure on Λ . Egan and Baez’s closure is on the ambient 27-dimensional lattice $\tilde{\Lambda} = \mathbb{Z}^3 \oplus \Lambda \subset \mathfrak{h}_3(\mathbb{O})$, not on Λ alone. For off-diagonal Hermitian matrices $X, Y \in \mathfrak{h}_3(\mathbb{O})$ representing $u = (x, y, z)$ and $v = (x', y', z')$ in $\Lambda \subset \mathbb{O}^3$, direct computation gives

$$2(XY + YX) = D + R,$$

where R is an off-diagonal residue lying in Λ and $D \in \mathbb{Z}^3$ is a diagonal piece whose entries are four times the block-pair Euclidean inner products of u and v :

$$D = 4(\langle x, x' \rangle + \langle y, y' \rangle, \langle x, x' \rangle + \langle z, z' \rangle, \langle y, y' \rangle + \langle z, z' \rangle),$$

so $\text{tr } D = 8\langle u, v \rangle$ recovers the \mathbb{R}^{24} inner product of u and v . Projecting onto the off-diagonal subspace defines a \mathbb{Z} -bilinear closed product $\varphi(u, v) := R$ on Λ — an order in the sense of Theorem 1.1, but obtained by projecting from a larger ambient with the inner-product shadow D discarded. For minimal Leech vectors $u \in \text{Min}(\Lambda)$, squaring gives $\text{tr } D = 8N(u) = 64$, so the projection is non-trivial on every minimal-vector input. The induced product φ is moreover commutative (the Jordan product is symmetric in its arguments), whereas \star is not (Section 5); the two orders are therefore distinct bilinear products on \mathbb{R}^{24} that both close on Λ , reaching that closure through different ambient algebras.

Earlier precursors. Further back, several constructions foreshadow the octonion-triple and Jordan-algebra frameworks above. Conway and Sloane [ConwaySloane1982] give 23 constructions of Λ , one for each Niemeier lattice, including a Turyn-style construction from three copies of E_8 . Lepowsky and Meurman [LepowskyMeurman1982] describe Λ as a sublattice of E_8^3 containing $2E_8^3$ and use this to generate the Conway group Co_0 via permutations of the three E_8 factors together with one extra automorphism. Elkies and Gross [ElkiesGross1996] obtain Λ from an integer form of the exceptional Jordan algebra $\mathfrak{h}_3(\mathbb{O})$ as the orthogonal complement of a distinguished identity element, and show that the automorphism group of this integer form is generated by octave reflections. Each of these constructions is additive or geometric: none defines or tests a bilinear product on Λ .

Kirmse (1924) [Kirmse1924] exhibited the E_8 lattice as a unimodular integral quadratic lattice inside the real octonion algebra. His lattice is not closed under multiplication, an error corrected by Coxeter [Coxeter1946] by swapping two basis indices in the Fano-plane labelling.

The present construction continues the same conceptual line, but the two transpositions in the chain are structurally different. Coxeter’s correction is a permutation of Fano *triples*: it rewrites the multiplication table without being induced by a linear map on \mathbb{R}^8 . The present σ is a permutation of *coordinate axes*: a linear involution of \mathbb{R}^8 that conjugates the multiplication, $\cdot \mapsto \cdot_\sigma$ (Definition 3.1; cf. Remark 3.3). Both modify the bilinear product, but each resolves a closure obstruction at a different stratum: Coxeter’s at the E_8 stratum, forcing $L \cdot L \subseteq L$; σ at a finer-grained sublattice stratum, forcing $\sigma(Ls) \cdot \sigma(Ls) \subseteq \sigma(Ls)$ (Lemma 4.4). Because $\sigma(L) = L$, the σ -twist preserves Coxeter’s closure automatically: $L \cdot_\sigma L = \sigma(L \cdot L) \subseteq L$. No choice of σ can return the construction to Kirmse’s pre-Coxeter regime.

7 Conclusion

The Leech lattice admits an order under a remarkably simple construction: a \mathbb{Z}_3 -symmetric triple-octonion product on \mathbb{R}^{24} , realised in the Coxeter representation aligned with Wilson’s sublattice description of Λ . The symbolic proof (Section 4) reduces closure to four lemmas, of which the crux is Lemma 4.4: the sublattice $\sigma(Ls)$ — unlike Ls itself — is closed under the standard octonion product. The twist maps the condition-3 sublattice to this closed sublattice, and the involutory nature of σ maps the result back.

Several questions remain immediately open:

- A structural reason for Lemma 4.4. The proof in Appendix A exhibits the 64 integer coefficients certifying $\sigma(Ls) \cdot \sigma(Ls) \subseteq \sigma(Ls)$, but does not explain *why* $\sigma(Ls)$ is closed under the standard octonion product while Ls is not. An identification of $\sigma(Ls)$ with a left ideal, with the image under a norm-preserving octonion automorphism, or with some module-theoretic structure over L , would replace the appendix tables by a structural argument.
- The automorphism group of $(\Lambda, +, \star)$ and its relationship to the Conway group Co_0 .
- Whether the order is maximal in any appropriate sense.
- Classification of the algebra $(\mathbb{R}^{24}, +, \star)$: it is non-associative, non-alternative, non-flexible, and non-unital (as an order), placing it outside the classical families.
- Whether other linear coordinate permutations of \mathbb{R}^8 (beyond simple transpositions of imaginary axes) conjugate the octonion multiplication into a triple product that closes on Λ in the sense of Theorem 1.1.

8 Outlook

8.1 A ternary reformulation

A more natural classification of $(\mathbb{R}^{24}, +, \star)$ may emerge from a *ternary* rather than binary viewpoint. The Leech lattice carries a prominent \mathbb{Z}_3 -symmetry (the cyclic permutation of the three octonion blocks), and \star is assembled from nine bilinear building blocks routed by that \mathbb{Z}_3 -action. Ternary composition algebras, as studied by Elduque [Elduque1996, Elduque2000_Triality, Elduque2023_IsotropicNorm]

and others [SmithVojtechovsky2022, KamiyaOkubo2015, Okubo1995_Book], are typically non-associative, non-commutative, non-alternative, and non-power-associative — the same negative classification $(\mathbb{R}^{24}, +, \star)$ satisfies — yet carry high internal symmetry, with order-3 automorphisms playing the central role: the Okubo algebra has automorphism group $SU(3)$, and its product is built from an order-3 automorphism of the octonions. Hall’s monograph [Hall2019_Moufang] establishes that Moufang loops and groups with triality are essentially the same object. Recasting \star as a trace-like reduction of a ternary operation compatible with the \mathbb{Z}_3 -block symmetry is a natural direction for further work.

8.2 Algebraic models in physics

Programmes that model fundamental physics on normed-division-algebra, triality, or symmetric-composition-algebra structures [Koeplinger2023, FureyHughes2025_TrioOfTriality] are one setting in which a non-classical bilinear product on \mathbb{R}^{24} may be of interest.

Acknowledgments

The author thanks Geoffrey M. Dixon and Robert A. Wilson for helpful discussions and advice on octonionic structures associated with the Leech lattice.

A Explicit basis tables for Lemmas 4.2–4.4

We fix concrete \mathbb{Z} -bases for the lattices L , $\sigma(Ls)$, and $\sigma(L\bar{s})$, and tabulate the integer coefficients of the basis-by-basis products called for in the proofs of Lemmas 4.2, 4.3, and 4.4. Each table has 64 entries and is arranged so that the row labelled (i, j) gives the unique 8-tuple $(c_1, \dots, c_8) \in \mathbb{Z}^8$ representing the corresponding product in the basis indicated. All entries are integer; closure under the standard octonion product follows by \mathbb{Z} -bilinearity. The same identities have been recomputed in two independent computer-algebra implementations as a cross-check (Appendix B).

Coordinates throughout are with respect to the basis $\{e_0, \dots, e_7\}$ and the standard Fano triples (2.1).

The basis $\{L_k\}$. Concretely: $L_1 = s = \frac{1}{2}(-e_0 + e_1 + \dots + e_7)$, $L_2 = e_0 + e_1$, $L_3 = e_0 - e_1$, and $L_{k+2} = e_0 + e_k$ for $k = 2, \dots, 6$. The eight vectors are linearly independent over \mathbb{Q} and span L over \mathbb{Z} .

$$\begin{aligned} L_1 &= \left(\frac{-1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \\ L_2 &= (1, 1, 0, 0, 0, 0, 0, 0) \\ L_3 &= (1, -1, 0, 0, 0, 0, 0, 0) \\ L_4 &= (1, 0, 1, 0, 0, 0, 0, 0) \\ L_5 &= (1, 0, 0, 1, 0, 0, 0, 0) \\ L_6 &= (1, 0, 0, 0, 1, 0, 0, 0) \\ L_7 &= (1, 0, 0, 0, 0, 1, 0, 0) \\ L_8 &= (1, 0, 0, 0, 0, 0, 1, 0) \end{aligned}$$

The basis $\{M_k\}$. Defined by $M_k := \sigma(L_k \cdot s)$ with $\sigma = (1\ 2)$ acting on coordinates by swapping e_1 and e_2 . Spans $\sigma(Ls)$ over \mathbb{Z} .

$$M_1 = \left(\frac{-3}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}\right)$$

$$M_2 = (-1, 0, 0, 0, 1, 0, 1, 1)$$

$$M_3 = (0, 1, 1, 1, 0, 1, 0, 0)$$

$$M_4 = (-1, 0, 1, 0, 0, 1, 0, 1)$$

$$M_5 = (-1, 1, 1, 0, 0, 0, 1, 0)$$

$$M_6 = (-1, 1, 0, 1, 0, 0, 0, 1)$$

$$M_7 = (-1, 0, 1, 1, 1, 0, 0, 0)$$

$$M_8 = (-1, 1, 0, 0, 1, 1, 0, 0)$$

The basis $\{N_k\}$. Defined by $N_k := \sigma(L_k \cdot \bar{s})$. Spans $\sigma(L\bar{s})$ over \mathbb{Z} .

$$N_1 = (2, 0, 0, 0, 0, 0, 0, 0)$$

$$N_2 = (0, 0, -1, 0, -1, 0, -1, -1)$$

$$N_3 = (-1, -1, 0, -1, 0, -1, 0, 0)$$

$$N_4 = (0, -1, -1, 0, 0, -1, 0, -1)$$

$$N_5 = (0, -1, -1, -1, 0, 0, -1, 0)$$

$$N_6 = (0, -1, 0, -1, -1, 0, 0, -1)$$

$$N_7 = (0, 0, -1, -1, -1, -1, 0, 0)$$

$$N_8 = (0, -1, 0, 0, -1, -1, -1, 0)$$

Lemma 4.2: $L \cdot L \subseteq L$.

i	j	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8
1	1	-1	-1	-1	0	0	0	0	0
1	2	0	-2	-2	1	1	0	1	0
1	3	2	2	2	-1	-1	0	-1	0
1	4	0	-2	-2	0	1	1	0	1
1	5	2	1	2	-1	-1	0	0	-1
1	6	0	-1	-2	0	0	0	1	1
1	7	2	1	2	0	-1	-1	-1	0
1	8	2	2	2	-1	0	-1	-1	-1
2	1	2	1	2	-1	-1	0	-1	0
2	2	0	1	-1	0	0	0	0	0
2	3	0	1	1	0	0	0	0	0
2	4	0	0	-1	1	0	1	0	0
2	5	2	3	3	-1	0	-1	-1	-1
2	6	0	1	0	-1	0	1	0	0
2	7	0	0	-1	0	0	0	1	1
2	8	0	1	0	0	0	0	-1	1
3	1	0	-1	-2	1	1	0	1	0
3	2	0	1	1	0	0	0	0	0
3	3	0	-1	1	0	0	0	0	0
3	4	0	0	1	1	0	-1	0	0
3	5	-2	-3	-3	1	2	1	1	1
3	6	0	-1	0	1	0	1	0	0
3	7	0	0	1	0	0	0	1	-1
3	8	0	-1	0	0	0	0	1	1
4	1	2	2	2	-1	-1	-1	0	-1
4	2	0	1	0	1	0	-1	0	0
4	3	0	-1	0	1	0	1	0	0
4	4	0	-1	-1	2	0	0	0	0
4	5	0	-1	-1	1	1	0	1	0
4	6	0	0	-1	1	0	1	0	0
4	7	0	0	0	1	-1	0	1	0
4	8	2	2	3	0	-1	-1	-1	0
5	1	0	-1	-2	1	0	0	0	1
5	2	-2	-2	-4	1	2	1	1	1
5	3	2	2	4	-1	0	-1	-1	-1
5	4	0	0	0	1	1	0	-1	0
5	5	0	-1	-1	0	2	0	0	0
5	6	0	-1	-1	0	1	1	0	1
5	7	0	-1	-1	1	1	0	1	0
5	8	0	0	0	0	1	-1	0	1
6	1	2	1	2	0	0	-1	-1	-1
6	2	0	0	-1	1	0	1	0	0
6	3	0	0	1	-1	0	1	0	0
6	4	0	-1	0	1	0	1	0	0
6	5	0	0	0	0	1	1	0	-1
6	6	0	-1	-1	0	0	2	0	0
6	7	2	2	3	-1	-1	0	0	-1
6	8	0	-1	-1	0	1	1	0	1
7	1	0	-1	-2	0	1	1	0	0
7	2	0	1	0	0	0	0	1	-1
7	3	0	-1	0	0	0	0	1	1
7	4	0	-1	-1	1	1	0	1	0
7	5	0	0	0	-1	1	0	1	0
7	6	-2	-3	-4	1	1	2	2	1
7	7	0	-1	-1	0	0	0	2	0
7	8	0	0	-1	0	0	0	1	1
8	1	0	-2	-2	1	0	1	1	0
8	2	0	0	-1	0	0	0	1	1
8	3	0	0	1	0	0	0	-1	1
8	4	-2	-3	-4	2	1	1	1	2
8	5	0	-1	-1	0	1	1	0	1
8	6	0	0	0	0	-1	1	0	1
8	7	0	-1	0	0	0	0	1	1
8	8	0	-1	-1	0	0	0	0	2

Tab. 1: Coefficients c_k such that $L_i \cdot L_j = \sum_{k=1}^8 c_k L_k$, for the bases defined above.

Lemma 4.3: $L \cdot \sigma(L\bar{s}) \subseteq \sigma(L\bar{s})$.

i	j	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8
1	1	-1	-1	-1	0	0	0	0	0
1	2	0	-2	-2	0	1	1	0	1
1	3	2	2	2	0	-1	-1	0	-1
1	4	0	-1	-2	0	1	0	1	0
1	5	2	1	2	-1	-1	0	0	-1
1	6	0	-2	-2	1	0	0	1	1
1	7	2	2	2	-1	-1	-1	-1	0
1	8	2	1	2	0	0	-1	-1	-1
2	1	2	2	2	-1	-1	-1	0	-1
2	2	-1	0	-2	0	0	1	0	1
2	3	1	1	2	0	0	0	-1	0
2	4	0	0	-1	1	0	0	0	1
2	5	0	0	-1	0	1	1	0	0
2	6	1	1	1	0	-1	1	-1	0
2	7	1	2	2	-1	-1	0	0	0
2	8	1	1	1	-1	0	0	-1	1
3	1	0	-2	-2	1	1	1	0	1
3	2	1	2	2	0	0	-1	0	-1
3	3	-1	-1	0	0	0	0	1	0
3	4	0	0	1	1	0	0	0	-1
3	5	0	0	1	0	1	-1	0	0
3	6	-1	-1	-1	0	1	1	1	0
3	7	-1	-2	-2	1	1	0	2	0
3	8	-1	-1	-1	1	0	0	1	1
4	1	2	1	2	-1	-1	0	-1	0
4	2	0	0	-1	1	0	1	0	0
4	3	0	0	1	1	0	-1	0	0
4	4	0	-1	-1	2	0	0	0	0
4	5	2	2	3	0	0	-1	-1	-1
4	6	0	-1	0	1	0	1	0	0
4	7	0	-1	-1	1	0	0	1	1
4	8	0	0	0	1	0	0	-1	1
5	1	0	-1	-2	1	0	0	0	1
5	2	0	1	0	0	1	0	-1	0
5	3	0	-1	0	0	1	0	1	0
5	4	-2	-3	-4	2	2	1	1	1
5	5	0	-1	-1	0	2	0	0	0
5	6	0	-1	-1	0	1	1	0	1
5	7	0	0	-1	0	1	0	1	0
5	8	0	0	0	0	1	-1	0	1
6	1	2	1	2	0	0	-1	-1	-1
6	2	1	2	1	-1	0	0	0	-1
6	3	-1	-1	-1	1	0	1	1	0
6	4	0	1	0	1	0	0	0	-1
6	5	0	0	0	0	1	0	1	-1
6	6	-1	-2	-3	1	1	2	1	0
6	7	1	1	1	0	-1	0	1	-1
6	8	1	1	1	0	0	0	0	0
7	1	0	-1	-2	0	1	1	0	0
7	2	-1	-1	-2	0	1	1	1	1
7	3	1	0	2	0	-1	0	0	0
7	4	0	-1	-1	1	0	1	1	0
7	5	0	-1	0	0	1	1	0	0
7	6	-1	-2	-2	0	0	2	1	1
7	7	1	0	1	-1	0	0	1	0
7	8	-1	-3	-3	1	1	1	1	2
8	1	0	-2	-2	1	0	1	1	0
8	2	-1	-1	-3	1	1	0	1	1
8	3	1	2	3	-1	-1	-1	0	0
8	4	0	0	0	1	0	-1	1	0
8	5	0	-1	-1	0	1	0	1	1
8	6	-1	-1	-2	1	0	1	1	1
8	7	-1	-1	-2	0	0	0	2	1
8	8	1	1	1	0	-1	-1	0	1

Tab. 2: Coefficients c_k such that $L_i \cdot N_j = \sum_{k=1}^8 c_k N_k$, for the bases defined above.

Lemma 4.4: $\sigma(Ls) \cdot \sigma(Ls) \subseteq \sigma(Ls)$.

i	j	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8
1	1	-1	1	1	0	0	0	0	0
1	2	0	0	2	0	-1	-1	0	-1
1	3	-2	-2	-4	0	1	1	0	1
1	4	0	1	2	-2	-1	0	-1	0
1	5	-2	-1	-2	1	-1	0	0	1
1	6	0	2	2	-1	0	-2	-1	-1
1	7	-2	-2	-2	1	0	1	-1	0
1	8	-2	-1	-2	0	0	1	1	-1
2	1	-2	-3	-2	0	1	1	0	1
2	2	2	-1	1	0	0	0	0	0
2	3	2	3	3	0	-2	-2	0	-2
2	4	2	1	2	-1	0	-1	0	-2
2	5	4	4	7	-2	-3	-2	-1	-2
2	6	0	-2	-1	1	0	-1	2	0
2	7	2	1	2	0	-1	-2	-1	0
2	8	2	1	3	1	-1	-1	-1	-2
3	1	0	1	0	0	-1	-1	0	-1
3	2	-2	-3	-5	0	2	2	0	2
3	3	2	1	1	0	0	0	0	0
3	4	-2	-3	-6	1	2	1	2	2
3	5	0	-2	-3	0	1	2	1	0
3	6	0	-2	-3	1	0	1	0	2
3	7	2	3	2	-2	-1	0	-1	0
3	8	2	1	1	-1	1	-1	-1	0
4	1	-2	-1	-2	-1	1	0	1	0
4	2	0	-2	-1	-1	0	1	0	2
4	3	4	4	5	-1	-2	-1	-2	-2
4	4	2	1	1	-2	0	0	0	0
4	5	2	2	3	-2	-2	1	-1	-1
4	6	2	1	2	-1	-2	-1	0	0
4	7	4	5	7	-3	-2	-2	-3	-1
4	8	2	0	2	-1	0	0	-1	-1
5	1	0	1	2	-1	-2	0	0	-1
5	2	-2	-5	-6	2	1	2	1	2
5	3	2	3	2	0	-1	-2	-1	0
5	4	0	-1	-2	0	0	-1	1	1
5	5	2	1	1	0	-2	0	0	0
5	6	0	-1	-1	2	-1	-1	0	1
5	7	0	0	-1	0	-1	0	-1	2
5	8	2	2	2	0	-1	-1	-2	-1
6	1	-2	-2	-2	1	0	-1	1	1
6	2	2	1	2	-1	0	-1	-2	0
6	3	2	3	2	-1	0	-1	0	-2
6	4	0	0	-1	-1	2	-1	0	0
6	5	2	2	2	-2	-1	-1	0	-1
6	6	2	1	1	0	0	-2	0	0
6	7	2	4	3	-1	-1	-2	-2	-1
6	8	4	5	7	-2	-1	-3	-2	-3
7	1	0	2	2	-1	-1	-1	-2	0
7	2	0	-2	-1	0	1	2	-1	0
7	3	0	-2	-3	2	1	0	1	0
7	4	-2	-4	-6	1	2	2	1	1
7	5	2	1	2	0	-1	0	-1	-2
7	6	0	-3	-2	1	1	0	0	1
7	7	2	1	1	0	0	0	-2	0
7	8	0	-1	-1	1	2	0	-1	-1
8	1	0	1	2	0	0	-1	-1	-2
8	2	0	-2	-2	-1	1	1	1	0
8	3	0	0	-2	1	-1	1	1	0
8	4	0	1	-1	-1	0	0	1	-1
8	5	0	-1	-1	0	-1	1	2	-1
8	6	-2	-4	-6	2	1	1	2	1
8	7	2	2	2	-1	-2	0	-1	-1
8	8	2	1	1	0	0	0	0	-2

Tab. 3: Coefficients c_k such that $M_i \cdot M_j = \sum_{k=1}^8 c_k M_k$, for the bases defined above.

B Research methodology

This research was conducted as a collaboration between a human researcher and an AI agent (Claude, Anthropic) over a series of directed prompts, governed by a written methodology (the “Manifesto”) established at the outset.

The human researcher (the author) brought the initial hypothesis — motivated by prior work on Okubo algebras and autotopies of normed composition algebras — established the research methodology, and maintained control of the inquiry through prompt-by-prompt review. The AI agent performed the computational work: implementing octonion algebras, Wilson’s Leech lattice construction, trial algebras, and closure tests, and contributed mathematical analysis.

Tests were written from first principles to independently verify every foundation: that octonion multiplication satisfies the composition property $N(xy) = N(x)N(y)$, that Wilson’s construction produces a lattice with the correct minimal shell of 196,560 vectors, and that the E_8 lattice is

indeed a maximal order. This independent verification is how we know that the properties stated in the literature are correctly implemented. Closure of the twisted product was first observed on over 12,000,000 random pairs of minimal vectors with zero failures, which prompted the search for a symbolic proof.

Symbolic proof and numeric validation. The proof of Theorem 1.1 given in Section 4 is symbolic: each of the four lemmas is established by exhibiting a finite list of integer coefficients (Appendix A), and the conclusion follows by \mathbb{Z} -bilinearity. No floating-point or sampled arithmetic enters the proof. Independently, the same basis-by-basis identities that constitute the appendix tables were recomputed in two separate computer-algebra implementations as a cross-check: a Python implementation using `fractions.Fraction` exact rationals (`python_project/src/symbolic_proof_checks.py`), and a GAP/LOOPS implementation using GAP’s native exact rational arithmetic (`gap_project/tests/test_lemma`). Both implementations reproduce the same integer coefficients displayed in the appendix. This redundancy is validation, not proof: it serves to detect transcription or implementation errors in the appendix tables, but the logical content of the proof rests on the tables themselves.

The methodology required:

- Every claim backed by a reference or computation.
- Every prompt logged, with actions recorded.
- Every conclusion independently verified.
- Forward-evolving corrections: no rewriting history; errors are corrected in subsequent entries.

The path to the result was not direct. Six trial algebra families were explored and ruled out before the transposition twist was identified in trial 007. The systematic methodology — understanding the anatomy of failure (which conditions fail, for which vector types, and why) before trying the next approach — was essential. The failure analysis of the untwisted product (Wilson’s condition (3) per Definition 2.3, exclusively type-3 \times type-3) directly pointed to the mechanism that the twist needed to correct.

Iterative convergence toward a robust result — including learning along the way what it is we are actually doing — is how study, research, and discovery have always proceeded. Mistakes were made on both sides: the AI agent occasionally departed from the agreed protocol or produced incorrect intermediate analyses, and the human author at times framed prompts under misconceptions or accepted intermediate results without sufficient scrutiny. In each case, the error was caught by the systematic cross-checking that the methodology requires — explicit verification, prompt-by-prompt review, and independent reimplementations in two computer-algebra systems — and corrected forward. What is novel here is the ability to trace the entire path, beginning to end, through the complete list of prompts.

The full record — all prompts, code, evidence, and git history — is available at the repository [repo]. The reader can form their own assessment of what this means for questions of discovery and attribution.

References

- [Baez2002] J. C. Baez, *The octonions*, Bull. Amer. Math. Soc. **39** (2002), 145–205. arXiv:math/0105155.
- [Baez2014] J. C. Baez (with contributions by G. Egan and others), *Integral octonions* (parts 1–12), The n -Category Café, November–December 2014. The material discussed here appears in parts 9 and 10: https://golem.ph.utexas.edu/category/2014/11/integral_octonions_part_9.html, https://golem.ph.utexas.edu/category/2014/12/integral_octonions_part_10.html.
- [ConwaySloane1982] J. H. Conway and N. J. A. Sloane, *Twenty-three constructions for the Leech lattice*, Proc. Roy. Soc. London Ser. A **381** (1982), 275–283.
- [ConwaySloane1999] J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups*, 3rd edition, Springer-Verlag, 1999.
- [ConwaySmith2003] J. H. Conway and D. A. Smith, *On Quaternions and Octonions: Their Geometry, Arithmetic, and Symmetry*, A. K. Peters, 2003.
- [Coxeter1946] H. S. M. Coxeter, *Integral Cayley numbers*, Duke Math. J. **13** (1946), 561–578.
- [Dixon1995] G. M. Dixon, *Octonions: invariant Leech lattice exposed*, preprint, 1995. arXiv:hep-th/9506080.
- [Dixon2010] G. M. Dixon, *Integral octonions, octonion XY -product, and the Leech lattice*, preprint, 2010. arXiv:1011.2541.
- [Elduque1996] A. Elduque, *On a class of ternary composition algebras*, J. Korean Math. Soc. **33** (1996), no. 1, 183–203.
- [Elduque2000_Triality] A. Elduque, *On triality and automorphisms and derivations of composition algebras*, Linear Algebra Appl. **314** (2000), no. 1–3, 49–74.
- [Elduque2023_IsotropicNorm] A. Elduque, *Okubo algebras with isotropic norm*, in: Non-Associative Algebras and Related Topics (NAART 2020), Springer Proc. Math. Stat., vol. 427, Springer, Cham, 2023, pp. 287–301. arXiv:2210.03456.
- [ElkiesGross1996] N. D. Elkies and B. H. Gross, *The exceptional cone and the Leech lattice*, Internat. Math. Res. Notices **1996**, no. 14, 665–698.
- [FureyHughes2025_TrioOfTriality] N. Furey and M. J. Hughes, *Three generations and a trio of trialities*, Phys. Lett. B **865** (2025), 139473. arXiv:2409.17948.
- [Hall2019_Moufang] J. I. Hall, *Moufang Loops and Groups with Triality are Essentially the Same Thing*, Memoirs of the American Mathematical Society, vol. 260, no. 1252, Amer. Math. Soc., Providence, RI, 2019.
- [KamiyaOkubo2015] N. Kamiya and S. Okubo, *Algebras satisfying triality and S_4 -symmetry*, preprint, 2015. arXiv:1503.00614.

- [Koeplinger2023] J. Köplinger, *Towards autotopies of normed composition algebras in algebraic quantum field theory*, preprint, 2023. DOI: 10.13140/RG.2.2.19003.39209.
- [LepowskyMeurman1982] J. Lepowsky and A. Meurman, *An E_8 -approach to the Leech lattice and the Conway group*, J. Algebra **77** (1982), no. 2, 484–504.
- [Okubo1995_Book] S. Okubo, *Introduction to Octonion and Other Non-Associative Algebras in Physics*, Montroll Memorial Lecture Series in Mathematical Physics, vol. 2, Cambridge University Press, Cambridge, 1995.
- [SmithVojtechovsky2022] J. D. H. Smith and P. Vojtěchovský, *Okubo quasigroups*, Doc. Math. **27** (2022), 535–580.
- [Kirmse1924] J. Kirmse, *Über die Darstellbarkeit natürlicher ganzer Zahlen als Summen von acht Quadraten und über ein mit diesem Problem zusammenhängendes nichtkommutatives und nichtassoziatives Zahlensystem*, Ber. Verh. Sächs. Akad. Wiss. Leipzig, Math.-Phys. Kl. **76** (1924), 63–82.
- [Pettersson2018] H. P. Petersson, *Integral octonions*, Lecture, Málaga Workshop on Non-Associative Algebras, 2018.
- [Wilson2009] R. A. Wilson, *Octonions and the Leech lattice*, J. Algebra **322** (2009), 2186–2190.
- [repo] J. Koeplinger, *An order on the Leech lattice from a \mathbb{Z}_3 -symmetric triple-octonion product: research repository*, https://bitbucket.org/jenskoeplinger/leech_alg, 2026.