A trilinear-irreducible order on the Leech lattice via the σ-twisted Okubo product

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Abstract

We construct a trilinear product T_{Okubo} on $\mathbb{R}^{24}$ under which the Leech lattice Λ is closed, and show that the resulting order is genuinely ternary. At its foundation lie the octonions $\mathbb{O}$, the unital 8-dimensional normed division algebra. Applied to them, the Petersson construction – a twist by an order-3 automorphism τ – yields the Okubo algebra, an 8-dimensional nonunital normed division algebra. Carrying it out with a σ-twisted octonion product in place of the standard one, σ a transposition of two imaginary units, yields the *σ-twisted Okubo product* $*_O$. Three copies of $*_O$, one on each octonion block of $\mathbb{R}^{24} = \mathbb{O}^3$, coupled by a $\mathbb{Z}_3$-symmetric cross-block routing, assemble the trilinear product T_{Okubo}. We prove that Λ is closed under T_{Okubo}, so that $(\Lambda, +, T_{\text{Okubo}})$ is a ternary order – an algebraic structure on Λ that is distinct from the binary order demonstrated in [Koeplinger2026_TripleO].

This new ternary order is *trilinear-irreducible*: when decomposing T_{Okubo}, it factors, cell by cell, into two nested applications of a binary block-product derived from $*_O$ – but that binary product does not itself close on Λ. A related rigidity is visible already in the per-block algebra: where the octonions and the Okubo algebra are highly symmetric – automorphism groups G_2 of dimension 14 and $SU(3)$ of dimension 8, respectively – the σ-twisted Okubo algebra $(\mathbb{O}, *_O)$ has automorphism group only $U(2)$, of dimension 4. That algebra is normed, hence a nonunital division algebra; but it is otherwise non-classical: like the Okubo algebra it is noncommutative, nonalternative, and not power-associative, and the σ-twist costs it, in addition, flexibility, third-power associativity, and the symmetric-composition identity.

We close by comparing the construction with Corradetti's recent study [Corradetti2026] of the Okubo product on the E_8-order.

This result was developed through a systematic human–AI collaboration; the complete research record is publicly available [repo].

1 Introduction

A companion paper [Koeplinger2026_TripleO] showed that the Leech lattice Λ [ConwaySloane1999] admits the structure of an *order* in a 24-dimensional real algebra: a bilinear product on $\mathbb{R}^{24}$ under which Λ is closed. That product – the $\mathbb{Z}_3$-symmetric triple-octonion product $\star$ – is binary. The present paper asks a different question: does Λ also carry a genuinely *ternary* order, one that is not a binary order in disguise?

We answer this affirmatively. The building block is the *σ-twisted Okubo product* $*_O$ on $\mathbb{O}$ (Definition 3.6); routing it over three octonion blocks by a cyclic $\mathbb{Z}_3$ template – the same template under which the companion paper's product is binary – gives a trilinear product T_{Okubo} on $\mathbb{R}^{24}$. The construction, and the per-block algebra it rests on, yield three results:

- **The σ-twisted Okubo algebra** (Section 4). The per-block product $(\mathbb{O}, *_O)$ – the Petersson construction of the Okubo algebra [Elduque1996, Elduque2023_IsotropicNorm] carried out with the σ-twisted octonion product – is a nonunital normed division algebra,[1] though *not* a symmetric composition algebra; it lies outside Elduque's classification, and its automorphism group is $U(2)$, strictly smaller than the $SU(3)$ of the Okubo algebra and the G_2 of the octonions.

- **A ternary order** (Theorem 5.7). Λ is closed under T_{Okubo}; hence $(\Lambda, +, T_{\text{Okubo}})$ is a ternary order.

- **Trilinear-irreducibility** (Theorem 6.3). T_{Okubo} admits a cell-level binary decomposition through a binary product $\circledast$ on $\mathbb{R}^{24}$, but $\circledast$ does *not* close on Λ. Thus the closure of Λ under T_{Okubo} does not factor through any binary order on Λ derived from $*_O$: the ternary order is genuinely trilinear, not a reassembled binary one.

The construction and the closure proof depend only on named results of the companion paper, summarized in Section 2, and on classical facts about integral octonions and composition algebras. Section 3 gives the construction; Section 4 characterizes the σ-twisted Okubo algebra; Section 5 proves closure; Section 6 proves trilinear-irreducibility.

2 The predecessor result

This section summarizes the results of the companion paper [Koeplinger2026_TripleO] that the present construction relies on.

Let $\mathbb{O}$ be the real octonion algebra with basis $\{e_0, e_1, \ldots, e_7\}$, e_0 the identity. Its multiplication is fixed by the seven Fano triples

$$(1,2,4), \quad (2,3,5), \quad (3,4,6), \quad (4,5,7), \quad (5,6,1), \quad (6,7,2), \quad (7,1,3), \tag{2.1}$$

each (a,b,c) encoding $e_a e_b = e_c$ – positive on cyclic, negative on anticyclic permutations – together with $e_i^2 = -e_0$ for $i \geq 1$. Conjugation is $\overline{e_0} = e_0$, $\overline{e_i} = -e_i$ $(i \geq 1)$, and the Hurwitz norm is $N(x) = x\,\overline{x}$. This is the Coxeter representation of the octonions used in the companion paper.

Wilson's embedding of Λ. Wilson [Wilson2009] embeds the Leech lattice in $\mathbb{O}^3 = \mathbb{R}^{24}$ as a sublattice of L^3, where $L \subset \mathbb{O}$ is the integer-octonion E_8 lattice, cut out by three coset conditions (restated precisely as Definition 3.2 below). The minimal shell $\mathrm{Min}(\Lambda)$ then has 196,560 vectors of squared norm 8.

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[1] Throughout, a *division algebra* is one with no zero divisors – equivalently, in finite dimension, one in which left and right multiplication by every nonzero element is bijective. A unit is not required, so a nonunital algebra may be a division algebra.

The σ-twist. The companion paper applies a transposition σ of two imaginary basis indices to the octonion product (2.1), yielding a second octonion product on the same $\mathbb{R}^8$ – the *σ-twisted octonion product* $\cdot_\sigma$ (Definition 3.1). The transposition is an algebra isomorphism, $\sigma : (\mathbb{O}, \cdot) \to (\mathbb{O}, \cdot_\sigma)$, and leaves both $\mathbb{O}$ and the integer-octonion lattice L invariant; its load-bearing effect for the closure of Λ is to move Wilson's sublattices $L \cdot \overline{s}$ and $L \cdot s$, on which the per-block product $\cdot_\sigma$ closes where the standard octonion product does not.

The binary triple-octonion order. The main result is that the $\mathbb{Z}_3$-symmetric triple-octonion product $\star$ on $\mathbb{R}^{24} = \mathbb{O}^3$ – the σ-twisted octonion product applied per block with $\mathbb{Z}_3$-symmetric cross-block routing – closes on Λ, so that $(\Lambda, +, \star)$ is an order in the real algebra $(\mathbb{R}^{24}, +, \star)$. This order is binary.

3 The construction

Definition 3.1 (σ-twisted product)**.** Let σ be a transposition of two imaginary basis indices.[2] The *σ-twisted octonion product* $\cdot_\sigma$ on $\mathbb{O}$ is the bilinear product whose multiplication table is obtained from the octonion table (2.1) by relabeling each Fano triple via σ. Equivalently, if $\rho_\sigma : \mathbb{O} \to \mathbb{O}$ is the $\mathbb{R}$-linear involution permuting basis vectors via σ, then

$$x \cdot_\sigma y := \rho_\sigma\big(\rho_\sigma(x) \cdot \rho_\sigma(y)\big), \tag{3.1}$$

where $\cdot$ is the standard octonion product.

Definition 3.2 (Wilson's octonionic Leech lattice)**.** Let $L \subset \mathbb{O}$ be the integer-octonion E_8 lattice [Coxeter1946, Petersson2018], the $\mathbb{Z}$-lattice spanned by the 240 roots

$$\{\pm e_a \pm e_b : 0 \le a < b \le 7\} \;\cup\; \{\tfrac{1}{2}(\pm e_0 \pm e_1 \pm \cdots \pm e_7) : \text{odd } \# \text{ of minus signs}\}. \tag{3.2}$$

Set $s := \frac{1}{2}(-e_0 + e_1 + e_2 + \cdots + e_7) \in L$. The overline $\overline{\cdot}$ denotes octonion conjugation and, applied elementwise, the induced involution on subsets of $\mathbb{O}$; thus $\overline{s}$ is the conjugate octonion and $\overline{L}$ the conjugate lattice. The conjugate lattice $\overline{L}$ is a rank-8 $\mathbb{Z}$-sublattice of $\mathbb{O}$ that shares its integer-coordinate part with L but whose half-integer roots have *even* number of minus signs (vs. L's odd). In particular $\overline{L} \neq L$ as subsets of $\mathbb{O}$; the two lattices coincide on the integer-coordinate sublattice $L \cap \overline{L} = \{c \in \mathbb{Z}^8 : \sum_i c_i \in 2\mathbb{Z}\}$ and are exchanged by $\overline{\cdot}$ on their half-integer parts. Wilson's octonionic Leech lattice $\Lambda \subset \mathbb{O}^3$ [Wilson2009] is the set of triples $(x, y, z) \in L^3$ satisfying

$$\text{(i) } x, y, z \in L; \quad \text{(ii) } x + y,\ x + z,\ y + z \in L \cdot \overline{s}; \quad \text{(iii) } x + y + z \in L \cdot s. \tag{3.3}$$

The minimal shell $\mathrm{Min}(\Lambda) \subset \Lambda$ at squared Euclidean norm 8 has $|\mathrm{Min}(\Lambda)| = 196{,}560$.

Remark 3.3 (Related octonionic route to Λ). Wilson's three-coset characterization is one of two independent octonionic routes to the Leech lattice; the other is Dixon's construction [Dixon1995, Dixon2010] via the octonion *XY-product* $A \circ_{XY} B := (AX)(\overline{Y}B)$. The present paper builds on Wilson's route only; the conjugate lattice $\overline{L}$ above is elementary (the image of L under octonion conjugation).

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[2] This σ is precisely the Coxeter–Bruck transposition of integral octonions; see [Koeplinger2026_TripleO, Appendix B] for the historical synthesis.

Definition 3.4 (Standard order-3 automorphism)**.** Let $\tau \in \mathrm{Aut}(\mathbb{O})$ be the order-3 automorphism cycling $e_1 \to e_2 \to e_4 \to e_1$ and $e_3 \to e_6 \to e_5 \to e_3$, fixing e_0 and e_7. The subgroup $\langle\tau\rangle \subset \mathrm{Aut}(\mathbb{O}) = G_2$ it generates is cyclic of order 3; τ and τ^2 are its two non-identity elements, and $\langle\tau\rangle$ lifts the standard order-3 triality automorphism of D_4 [Elduque2000_Triality].

Remark 3.5. The two non-identity elements τ, τ^2 of $\langle\tau\rangle \cong \mathbb{Z}/3$ are the cube roots of the identity *within this subgroup* – both satisfy $\tau^3 = \mathrm{id}$ and neither is the identity. This is not a count of all order-3 elements of G_2 (of which there are many); the pair $\{\tau, \tau^2\}$ is singled out as the generators of the specific $\mathbb{Z}/3$ that lifts D_4-triality. Fixing τ rather than τ^2 is a labelling convention.

Definition 3.6 (The σ-twisted Okubo product)**.** The *σ-twisted Okubo product* $*_O$ on $\mathbb{O}$ is defined by

$$x *_O y \;:=\; \tau(\overline{x}) \cdot_\sigma \tau^2(\overline{y}). \tag{3.4}$$

It is the Petersson construction $(x, y) \mapsto \tau(\overline{x}) \cdot \tau^2(\overline{y})$ of the Okubo algebra, carried out with the σ-twisted product $\cdot_\sigma$ in place of the standard octonion product $\cdot$. The algebra $(\mathbb{O}, *_O)$ is characterized in Section 4. For the Petersson construction and the Okubo algebra, see [Petersson1969, Okubo1995_Book, Elduque1996, Elduque2023_IsotropicNorm, KamiyaOkubo2015, SmithVojtechovsky2022, MarraniCorradettiZucconi2025].

Definition 3.7 (Cyclic $\mathbb{Z}_3$ routing)**.** For $\gamma \in \{1, 2, 3\}$ set

$$\mathcal{C}_\gamma \;:=\; \{(\alpha, \beta, \rho) \in \{1, 2, 3\}^3 : \alpha + \beta + \rho + \gamma \equiv 0 \pmod 3\}. \tag{3.5}$$

Each $\mathcal{C}_\gamma$ has $|\mathcal{C}_\gamma| = 9$, with $\bigsqcup_\gamma \mathcal{C}_\gamma = \{1, 2, 3\}^3$.

Definition 3.8 (The ternary product T_{Okubo})**.** For $X = (X_1, X_2, X_3)$, $Y = (Y_1, Y_2, Y_3)$, $Z = (Z_1, Z_2, Z_3) \in \mathbb{O}^3$,

$$T_{\mathrm{Okubo}}(X, Y, Z)_\gamma \;:=\; \sum_{(\alpha,\beta,\rho)\in\mathcal{C}_\gamma} X_\alpha *_O \big(Y_\beta *_O Z_\rho\big), \qquad \gamma \in \{1, 2, 3\}. \tag{3.6}$$

The product T_{Okubo} is $\mathbb{R}$-trilinear, since $*_O$ is bilinear.

4 The σ-twisted Okubo algebra

The per-block product $*_O$ (Definition 3.6) drives the whole construction. This section characterizes the algebra $(\mathbb{O}, *_O)$ in its own right. It remains *normed*: the norm N is anisotropic and multiplicative under $*_O$, so $(\mathbb{O}, *_O)$ is a nonunital *division* algebra. It is, however, non-classical – not symmetric, not flexible, not power-associative – and its automorphism group $U(2)$ is strictly smaller than that of any standard 8-dimensional composition algebra. All identity and automorphism computations below were carried out in exact arithmetic; the scripts are in the research repository [repo].

4.1 Isotope decomposition and identities

By Definition 3.6, $*_O$ factors as

$$x *_O y \;=\; \cdot_\sigma\big(A(x),\, B(y)\big), \qquad A := \tau \circ \overline{\cdot}, \quad B := \tau^2 \circ \overline{\cdot}, \tag{4.1}$$

exhibiting $(\mathbb{O}, *_O)$ as a *principal isotope* of the σ-twisted octonions $(\mathbb{O}, \cdot_\sigma)$, itself isomorphic to the octonion algebra through the relabeling ρ_σ. The maps A, B are isometries of the norm N – τ permutes the basis and $\overline{\cdot}$ negates the imaginary part – so $(\mathbb{O}, *_O)$ is a composition algebra: $N(x *_O y) = N(x)\,N(y)$.

The maps A, B are, however, not *automorphisms* of $(\mathbb{O}, \cdot_\sigma)$ – indeed τ is not an automorphism of $(\mathbb{O}, \cdot_\sigma)$ – so the isotope inherits none of the further identities of its base. Table 1 records which standard identities hold. For the power identities we distinguish *third-power associativity*, the single identity $x(xx) = (xx)x$, from *fourth-power associativity*, the agreement of all five bracketings of x^4; an algebra is *power-associative* when both hold.

An $\mathbb{R}$-linear involution $\iota : \mathbb{O} \to \mathbb{O}$ – one with $\iota^2 = \mathrm{id}$ – is a *canonical conjugate* for the algebra $(\mathbb{O}, p)$ if $x\,\iota(x) = \iota(x)\,x = N(x)\,e_0$ for every $x \in \mathbb{O}$ (the product being that of the algebra). The Petersson form (3.4) also distinguishes a pair of *side-specific conjugates*

$$\iota_L(x) := \tau^2(\overline{x}), \qquad \iota_R(x) := \tau(\overline{x}), \tag{4.2}$$

the maps that satisfy $x *_O \iota_L(x) = \iota_R(x) *_O x = N(x)\,e_0$. They are mutual inverses, $\iota_L \circ \iota_R = \iota_R \circ \iota_L = \mathrm{id}$, but not individually involutions ($\iota_L^2 = \tau$, $\iota_R^2 = \tau^2$). The e_0 here is the identity of $\mathbb{O}$ (Section 2), and of $\cdot_\sigma$. The algebra $(\mathbb{O}, *_O)$ is nonunital; e_0 appears in these identities not as an $*_O$-identity (there is none) but as the identity of the underlying octonion product, the choice of which determines ι_L and ι_R.

property	octonions	para-octonions	Okubo	$(\mathbb{O}, *_O)$
automorphism group	G_2	G_2	$SU(3)$	$U(2)$
normed division algebra	✓	✓	✓	✓
composition $N(xy) = N(x)N(y)$	✓	✓	✓	✓
symmetric composition $(xy)x = N(x)\,y$	–	✓	✓	–
admits a Z/2 canonical conjugate	✓	✓	–	–
$\iota_L \circ \iota_R = \mathrm{id}$	✓	✓	✓	✓
$x\,\iota_L(x) = N(x)\,e_0$	–	–	✓	✓
$\iota_R(x)\,x = N(x)\,e_0$	–	–	✓	✓
flexibility $x(yx) = (xy)x$	✓	✓	✓	–
alternativity $x(xy) = (xx)y$, $(yx)x = y(xx)$	✓	–	–	–
third-power assoc. $x(xx) = (xx)x$	✓	✓	✓	–
power-associative	✓	–	–	–
unital	✓	–	–	–

Tab. 1: Properties satisfied (✓) or failed (–) by the four real 8-dimensional algebras built on $\mathbb{O}$: the octonions, the para-octonions $x \bullet y = \overline{x}\,\overline{y}$, the Okubo algebra, and the σ-twisted Okubo algebra $(\mathbb{O}, *_O)$. The first row records the automorphism group; for $(\mathbb{O}, *_O)$ this is the identity component (Proposition 4.1).

Composition holds for $(\mathbb{O}, *_O)$ by (4.1). Each failure is witnessed by an explicit pair of basis vectors: at $x = e_1$, $y = e_3$,

$$x *_O (y *_O x) = -e_6 \;\neq\; -e_7 = (x *_O y) *_O x,$$

which breaks flexibility; the same pair gives $(x *_O y) *_O x = -e_7 \neq e_3 = N(x)\,y$, breaking symmetric composition; alternativity fails likewise, and $*_O$ fails even third-power associativity, hence is not power-associative [repo]. The para-octonions and the Okubo algebra behave differently on the power identities: as symmetric composition algebras they satisfy the cubic identity $x(xx) = (xx)x = N(x)\,x$ and so are third-power associative, but already $(xx)(xx) \neq ((xx)x)x$, so neither is fourth-power associative; among the four algebras only the octonions, being alternative, are power-associative. A two-sided identity element would solve a linear system that has no solution, so $(\mathbb{O}, *_O)$ is nonunital. It is nonetheless a *division* algebra: the norm N is positive-definite, hence anisotropic, and the composition law $N(x *_O y) = N(x)\,N(y)$ then leaves $*_O$ no zero divisors, so left and right multiplication by any nonzero element is bijective. It thus lies outside Elduque's classification of the symmetric composition algebras – the para-Hurwitz and Okubo families [Elduque1996, Elduque2000_Triality, Elduque2023_IsotropicNorm] – and is, by (4.1), a *non-symmetric composition-algebra isotope*, the isotope of the σ-twisted octonions by the two non-automorphism isometries A, B.

4.2 The automorphism group

Proposition 4.1. *The automorphism group* $\mathrm{Aut}(\mathbb{O}, *_O)$ *is compact, with identity component*

$$\mathrm{Aut}(\mathbb{O}, *_O)^0 \;\cong\; U(2),$$

the group of 2×2 *complex unitary matrices.*

Proof (by exact computation; scripts in [repo]). The derivation algebra $\mathfrak{Der} = \{D \in \mathrm{End}_{\mathbb{R}}(\mathbb{O}) : D(x *_O y) = D(x) *_O y + x *_O D(y)\}$ is the solution space of a homogeneous linear system. It is 4-dimensional, and every solution is skew-symmetric, so $\mathfrak{Der} \subseteq \mathfrak{so}(8)$ and $\mathrm{Aut}(\mathbb{O}, *_O)$ is compact. The derived algebra $[\mathfrak{Der}, \mathfrak{Der}]$ is 3-dimensional with negative-definite Killing form and the center is 1-dimensional, so $\mathfrak{Der} \cong \mathfrak{su}(2) \oplus \mathfrak{u}(1) = \mathfrak{u}(2)$. The Casimir element of the $\mathfrak{su}(2)$-summand has eigenvalues $\frac{3}{8}$ and 0 on $\mathbb{O}$, so $\mathbb{O}$ splits as a quaternionic spin-$\frac{1}{2}$ representation (dimension 4) and a trivial one (dimension 4); the half-integer spin forces the simple factor to be the simply-connected $SU(2)$. The generator C of the $\mathfrak{u}(1)$-center satisfies $\exp\bigl((2\pi/\sqrt{3})\,C\bigr) = -\mathrm{id}$ on the spin-$\frac{1}{2}$ block and $+\mathrm{id}$ on the trivial block – the central element $-\mathrm{id}$ of $SU(2)$. So the $SU(2)$- and $U(1)$-factors meet in a $\mathbb{Z}_2$, and $\mathrm{Aut}(\mathbb{O}, *_O)^0 = (SU(2) \times U(1))/\mathbb{Z}_2 = U(2)$. □

The discrete data are consistent with this: of the 256 sign-changes of the basis (the diagonal maps $e_i \mapsto \pm e_i$), only the identity and one other are automorphisms, and that other lies inside $U(2)$; no basis permutation, nor τ nor $\bar{\cdot}$, is an automorphism. Proposition 4.1 places $(\mathbb{O}, *_O)$ at the foot of the chain

$$U(2) \;\subset\; SU(3) \;\subset\; G_2, \qquad \dim 4 < \dim 8 < \dim 14,$$

of automorphism groups of the σ-twisted Okubo algebra, the Okubo algebra, and the octonions: the σ-twist strictly reduces the symmetry. This rigidity fits the role $*_O$ plays in the closure of Λ: the Okubo product, whose automorphism group is the larger $SU(3)$, does *not* close Λ under the $\mathbb{Z}_3$ routing [repo], whereas the rigid $*_O$ does (Theorem 5.7).

4.3 Comparison with Corradetti's integral Okubo elements

A recent paper of Corradetti [Corradetti2026] studies the arithmetic of the Okubo product on the integral octonions, the lattice L of Definition 3.2 (which Corradetti calls the "Coxeter–Dickson E_8-order"). Corradetti shows that the Okubo product itself does not preserve this order: written in an integral octonion basis it is defined over the ring $\mathbb{Z}[\sqrt{3}]$, and generates a $\mathbb{Z}[\sqrt{3}]$-order rather than a $\mathbb{Z}$-order. The lattice E_8 is recovered from that $\mathbb{Z}[\sqrt{3}]$-order by a 2-adic saturation. The $\sqrt{3}$ enters with the order-3 automorphism of the Petersson construction: a generic order-3 automorphism of the octonions is a rotation, and its real coordinates carry the $\sqrt{3}$ of $\cos\frac{2\pi}{3} = -\frac{1}{2}$ and $\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$.

The present paper takes a parallel route on the same lattice L. Its order-3 automorphism τ is a basis *permutation* (Definition 3.4) rather than a rotation, so no $\sqrt{3}$ arises: the σ-twisted Okubo product is defined over $\mathbb{Z}$, and $*_O(L, L) \subseteq \overline{L}$ (Lemma 5.1). The two routes pay a price in different places. Corradetti keeps the Okubo algebra and pays in the ring, passing through $\mathbb{Z}[\sqrt{3}]$; the present paper keeps the ring rational and pays in the algebra – a permutation cannot be a generic order-3 automorphism, so $(\mathbb{O}, *_O)$ is not the Okubo algebra but the non-symmetric isotope characterized earlier in this section. They are two integralizations of an Okubo-type composition product on E_8.

The present paper's distinctive step is the lift to dimension 24. Corradetti's route and the per-block product $*_O$ both reach E_8 in dimension 8; routed over the three octonion blocks, the ternary product T_{Okubo} reaches the Leech lattice in dimension $24 = 3 \cdot 8$ (Theorem 5.7). The two targets are distinguished: E_8 is the unique even unimodular lattice of dimension 8, and the Leech lattice the unique even unimodular lattice of dimension 24 with no roots. A third independent appearance of this 2-primary signature lies in the companion paper, where Wilson's identity $L \cdot \overline{s} \cap L \cdot s = 2L$ descends the octonion product on L to an $\mathbb{F}_2$-bilinear product on $L/2L$ under which the Wilson sublattices project to a complementary pair of Lagrangians (two-sided ideal and subalgebra) [Koeplinger2026_TripleO, Appendix A.1]. An Okubo-type composition product has thus reached, in each dimension, the lattice that even unimodular uniqueness alone would pin down. Whether that is a coincidence of separate constructions, or the trace of a single integralization mechanism – one that, once known to yield an even unimodular lattice (rootless, in dimension 24), can reach no other target – is not settled here. We have not excluded a common provenance; we record the parallel as an open question, and the relation between $(\mathbb{O}, *_O)$ and the integral Okubo algebra as the place it would be settled.

5 Closure of Λ under T_{Okubo}

This section proves that Λ is closed under T_{Okubo} (Theorem 5.7). The proof is structural: T_{Okubo} carries each of Wilson's three coset conditions through a clean inclusion of the σ-twisted Okubo product with the relevant coset lattice (Lemma 5.6), the three inclusions being assembled by the $\mathbb{Z}_3$ routing. Those Okubo inclusions are in turn reduced – through the isotopy identity that presents $*_O$ as a twist of the σ-twisted octonion product $\cdot_\sigma$ (Lemma 5.4) – to coset inclusions of $\cdot_\sigma$: one is a closure lemma of the predecessor paper [Koeplinger2026_TripleO], the other three are finite 64-entry structure-constant tables (Lemma 5.5). The same closure statement also follows from an independent finite reduction (Lemma 5.8 and Remark 5.9). All finite verifications in this section are carried out for the transposition $\sigma = (1, 2)$ (Definition 3.1).

5.1 Lemmas

Six lemmas feed the closure proof. Two are bookkeeping – the per-block image of $*_O$ and the linearity of the routed total. The next two are the structural inputs – the action of τ and conjugation on Wilson's cosets (used again in Section 6), and the isotopy identity presenting $*_O$ through $\cdot_\sigma$. The last two are the heart of the matter: the coset inclusions of $\cdot_\sigma$, and the three Okubo coset inclusions assembled from them.

Lemma 5.1 (Per-block image of $*_O$ on L)**.** $*_O(L, L) \subseteq \overline{L}$.

Proof. By Definition 3.6, $x *_O y = \tau(\overline{x}) \cdot_\sigma \tau^2(\overline{y})$. For $x, y \in L$ we have $\overline{x}, \overline{y} \in \overline{L}$ by the definition of $\overline{L}$. The automorphism τ permutes the basis $\{e_0, \ldots, e_7\}$, leaving the integer-coordinate sublattice $L \cap \overline{L}$ and the half-integer parity class invariant; hence τ and τ^2 preserve $\overline{L}$, and $\tau(\overline{x}), \tau^2(\overline{y}) \in \overline{L}$.

The product $\cdot_\sigma$ is closed on L: ρ_σ permutes the basis $\{e_0, \ldots, e_7\}$ and so preserves L (it fixes the parity class of (3.2)), while L is a ring under the standard octonion product [Coxeter1946]; hence $L \cdot_\sigma L = \rho_\sigma(\rho_\sigma L \cdot \rho_\sigma L) = \rho_\sigma(L \cdot L) \subseteq \rho_\sigma L = L$. Since ρ_σ commutes with conjugation, $\cdot_\sigma$ inherits the anti-isomorphism identity $\overline{a \cdot_\sigma b} = \overline{b} \cdot_\sigma \overline{a}$ from the standard $\overline{a \cdot b} = \overline{b} \cdot \overline{a}$. Conjugating both sides of $L \cdot_\sigma L \subseteq L$ gives $\overline{L} \cdot_\sigma \overline{L} = \overline{L \cdot_\sigma L} \subseteq \overline{L}$. Therefore $\tau(\overline{x}) \cdot_\sigma \tau^2(\overline{y}) \in \overline{L} \cdot_\sigma \overline{L} \subseteq \overline{L}$. ☐

Lemma 5.2 (Linearity of the routed total)**.** *For $X = (X_1, X_2, X_3) \in \mathbb{O}^3$ write $X_+ := X_1 + X_2 + X_3$ for its block sum, and similarly Y_+, Z_+. Then for all $X, Y, Z \in \mathbb{O}^3$,*

$$\sum_{\gamma=1}^{3} T_{\text{Okubo}}(X, Y, Z)_\gamma \;=\; X_+ *_O (Y_+ *_O Z_+). \tag{5.1}$$

Proof. By Definition 3.8, $\sum_\gamma T_{\text{Okubo}}(X, Y, Z)_\gamma$ ranges over $\bigsqcup_\gamma \mathcal{C}_\gamma = \{1, 2, 3\}^3$, so

$$\sum_\gamma T_{\text{Okubo}}(X, Y, Z)_\gamma = \sum_{(\alpha,\beta,\rho)\in\{1,2,3\}^3} X_\alpha *_O (Y_\beta *_O Z_\rho). \tag{5.2}$$

By bilinearity of $*_O$ in each slot the triple sum factors:

$$\sum_\alpha \sum_\beta \sum_\rho X_\alpha *_O (Y_\beta *_O Z_\rho) = \sum_\alpha X_\alpha *_O \Big(\sum_\beta Y_\beta *_O \sum_\rho Z_\rho\Big) = X_+ *_O (Y_+ *_O Z_+). \tag{5.3}$$

☐

Lemma 5.3 (Action of τ and conjugation on Wilson's cosets)**.** *$\tau(s) = s$ and $\tau(\overline{s}) = \overline{s}$; consequently $\tau(L \cdot s) = L \cdot s$ and $\tau(L \cdot \overline{s}) = L \cdot \overline{s}$, and the analogous identities hold with $\overline{L}$ in place of L. Moreover*

$$\overline{L \cdot s} = \overline{s} \cdot \overline{L}, \qquad \overline{L \cdot \overline{s}} = s \cdot \overline{L}, \tag{5.4}$$

with $\overline{L}$ – not L – on the right.

Proof. Write $s = \frac{1}{2}(-e_0 + \sum_{i=1}^{7} e_i)$. Since τ fixes e_0, e_7 and permutes $\{e_1, e_2, e_4\}$ and $\{e_3, e_5, e_6\}$, the sum $\sum_{i=1}^{7} e_i$ is τ-invariant, so $\tau(s) = s$ and $\tau(\overline{s}) = \overline{s}$ by linearity. τ preserves L and $\overline{L}$ (Lemma 5.1 proof), so $\tau(L \cdot s) = \tau(L) \cdot \tau(s) = L \cdot s$, and likewise for the other three cases. For (5.4), the anti-isomorphism $\overline{x \cdot y} = \overline{y} \cdot \overline{x}$ gives $\overline{L \cdot s} = \overline{s} \cdot \overline{L}$ and $\overline{L \cdot \overline{s}} = \overline{\overline{s}} \cdot \overline{L} = s \cdot \overline{L}$. ☐

Lemma 5.4 (Isotopy identity for $*_O$)**.** *Let $M, N \subseteq \mathbb{O}$ be τ-invariant, $\tau(M) = M$ and $\tau(N) = N$. Writing $*_O(M, N) := \{x *_O y : x \in M,\ y \in N\}$ and $\cdot_\sigma(N, M)$ likewise,*

$$*_O(M, N) \;=\; \overline{\cdot_\sigma(N, M)}\,. \tag{5.5}$$

This applies in particular to each of the six cosets L, $\overline{L}$, $L \cdot s$, $L \cdot \overline{s}$, $s \cdot \overline{L}$, $\overline{s} \cdot \overline{L}$, all τ-invariant by Lemma 5.3.

Proof. By Definition 3.6, $x *_O y = \tau(\overline{x}) \cdot_\sigma \tau^2(\overline{y})$. The automorphism τ fixes $e_0 = 1$, hence commutes with conjugation: $\tau(\overline{x}) = \overline{\tau(x)}$. As x ranges over M, $\tau(x)$ ranges over $\tau(M) = M$, so $\tau(\overline{x})$ ranges over $\overline{M}$; likewise $\tau^2(\overline{y})$ ranges over $\overline{N}$. Hence $*_O(M, N) = \cdot_\sigma(\overline{M}, \overline{N})$. The product $\cdot_\sigma$ inherits the conjugation anti-isomorphism $\overline{a \cdot_\sigma b} = \overline{b} \cdot_\sigma \overline{a}$ (proof of Lemma 5.1), so for $a \in N$, $b \in M$ one has $\overline{b} \cdot_\sigma \overline{a} = \overline{a \cdot_\sigma b}$; as a, b range over $N \times M$ this gives $\cdot_\sigma(\overline{M}, \overline{N}) = \overline{\cdot_\sigma(N, M)}$. ☐

Lemma 5.5 (Coset inclusions for $\cdot_\sigma$)**.** *The σ-twisted product satisfies*

$$\cdot_\sigma(\overline{L},\ L) \subseteq \overline{L}, \tag{5.6}$$

$$\cdot_\sigma(L \cdot \overline{s},\ L) \subseteq s \cdot \overline{L}, \tag{5.7}$$

$$\cdot_\sigma(\overline{s} \cdot \overline{L},\ L \cdot s) \subseteq \overline{s} \cdot \overline{L}. \tag{5.8}$$

Proof. $\cdot_\sigma$ is $\mathbb{R}$-bilinear, so each left-hand side is $\mathbb{Z}$-bilinear in its two lattice arguments. An inclusion $\cdot_\sigma(A, B) \subseteq C$ between rank-8 $\mathbb{Z}$-lattices therefore holds on all of $A \times B$ as soon as it holds on a pair of $\mathbb{Z}$-bases: every element of A, resp. B, is an integer combination of basis vectors, $\cdot_\sigma$ expands by bilinearity, and C is a $\mathbb{Z}$-module. Each of the three inclusions is thus equivalent to a 64-entry check – the 8×8 products of basis vectors, expressed in a $\mathbb{Z}$-basis of the target lattice, must have integer coordinates. The three 64-entry coefficient tables were computed in exact arithmetic; all 192 entries are integral, with zero failures [repo]. Appendix A records the data – the σ-twisted product and the lattice bases – from which this verification can be reproduced. ☐

Lemma 5.6 (Okubo coset inclusions)**.** *The σ-twisted Okubo product satisfies the three inclusions*

$$*_O(L,\ \overline{L}) \subseteq L, \tag{5.9}$$

$$*_O\big(L,\ *_O(L,\ L \cdot \overline{s})\big) \subseteq L \cdot \overline{s}, \tag{5.10}$$

$$*_O\big(L \cdot s,\ *_O(L \cdot s,\ L \cdot s)\big) \subseteq L \cdot s. \tag{5.11}$$

Proof. All six cosets are τ-invariant (Lemma 5.3), so the isotopy identity (5.5) applies to each product below.

Inclusion (5.9). By (5.5), $*_O(L, \overline{L}) = \overline{\cdot_\sigma(\overline{L}, L)}$. By (5.6), $\cdot_\sigma(\overline{L}, L) \subseteq \overline{L}$; applying conjugation gives $*_O(L, \overline{L}) \subseteq \overline{\overline{L}} = L$.

Inclusion (5.10). By (5.5), $*_O(L, L \cdot \overline{s}) = \overline{\cdot_\sigma(L \cdot \overline{s}, L)}$; by (5.7) $\cdot_\sigma(L \cdot \overline{s}, L) \subseteq s \cdot \overline{L}$, so $*_O(L, L \cdot \overline{s}) \subseteq \overline{s \cdot \overline{L}} = L \cdot \overline{s}$ by (5.4). Thus the inner product of (5.10) already lies in $L \cdot \overline{s}$, and a second application of the inclusion just proved gives $*_O\big(L, *_O(L, L \cdot \overline{s})\big) \subseteq *_O(L, L \cdot \overline{s}) \subseteq L \cdot \overline{s}$.

Inclusion (5.11). For the inner product, (5.5) gives $*_O(L \cdot s, L \cdot s) = \overline{\cdot_\sigma(L \cdot s, L \cdot s)}$. The predecessor paper proves $\cdot_\sigma(L \cdot s, L \cdot s) \subseteq L \cdot s$ – via $\cdot_\sigma(A, B) = \rho_\sigma(\rho_\sigma A \cdot \rho_\sigma B)$ this is the closure $\rho_\sigma(L \cdot s) \cdot \rho_\sigma(L \cdot s) \subseteq \rho_\sigma(L \cdot s)$ established there [Koeplinger2026_TripleO] – so $*_O(L \cdot s, L \cdot s) \subseteq \overline{L \cdot s} = \overline{s} \cdot \overline{L}$ by (5.4). For the outer product, (5.5) gives $*_O(L \cdot s,\ \overline{s} \cdot \overline{L}) = \overline{\cdot_\sigma(\overline{s} \cdot \overline{L}, L \cdot s)}$, and by (5.8) $\cdot_\sigma(\overline{s} \cdot \overline{L}, L \cdot s) \subseteq \overline{s} \cdot \overline{L}$, whence $*_O(L \cdot s, \overline{s} \cdot \overline{L}) \subseteq \overline{\overline{s} \cdot \overline{L}} = L \cdot s$ by (5.4). Combining, $*_O\big(L \cdot s, *_O(L \cdot s, L \cdot s)\big) \subseteq *_O(L \cdot s, \overline{s} \cdot \overline{L}) \subseteq L \cdot s$. ☐

5.2 The closure theorem

Closure is proved structurally: each of Wilson's three coset conditions (3.3) on the output triple $T_{\text{Okubo}}(X, Y, Z)$ is obtained from the corresponding condition on the inputs by one of the Okubo coset inclusions of Lemma 5.6, assembled through the $\mathbb{Z}_3$ routing. We then record an independent finite reduction that proves the same statement without any structural input.

Theorem 5.7 (Closure)**.** *$T_{\mathrm{Okubo}}(X,Y,Z) \in \Lambda$ for every $X,Y,Z \in \Lambda$. Consequently $(\Lambda, +, T_{\mathrm{Okubo}})$ is a ternary order in $(\mathbb{R}^{24}, +, T_{\mathrm{Okubo}})$.*

Proof. Fix $X,Y,Z \in \Lambda$ and put $W := T_{\mathrm{Okubo}}(X,Y,Z)$. We verify Wilson's three membership conditions (3.3) for the triple $W = (W_1, W_2, W_3)$, in their natural order.

Condition (i): each $W_\gamma \in L$. By Definition 3.8, W_γ is the sum of the nine terms $X_\alpha *_O (Y_\beta *_O Z_\rho)$ with $(\alpha,\beta,\rho) \in \mathcal{C}_\gamma$. Each $X_\alpha, Y_\beta, Z_\rho \in L$ by Wilson's condition (i) on X,Y,Z. By Lemma 5.1, $Y_\beta *_O Z_\rho \in \overline{L}$; hence $X_\alpha *_O (Y_\beta *_O Z_\rho) \in *_O(L, \overline{L}) \subseteq L$ by inclusion (5.9). Since L is closed under addition, the nine-term sum W_γ lies in L.

Condition (ii): the pair sums $W_\alpha + W_\beta$ lie in $L \cdot \overline{s}$. Fix $\{\alpha,\beta\} \subset \{1,2,3\}$ with $\alpha \neq \beta$. For each $(a,b) \in \{1,2,3\}^2$ the routing set $\mathcal{C}_\gamma$ contains exactly one cell with first two indices (a,b), namely $(a,b,c_\gamma(a,b))$ with $c_\gamma(a,b) \equiv -\gamma - a - b \pmod 3$; for $\gamma \in \{\alpha,\beta\}$ the two third indices $c_\alpha(a,b)$, $c_\beta(a,b)$ differ by $\beta - \alpha \not\equiv 0 \pmod 3$ and so are distinct. Bilinearity of $*_O$ in its third slot therefore gives

$$W_\alpha + W_\beta = \sum_{(a,b)\in\{1,2,3\}^2} X_a *_O \big(Y_b *_O z_{\alpha\beta}(a,b)\big), \qquad z_{\alpha\beta}(a,b) := Z_{c_\alpha(a,b)} + Z_{c_\beta(a,b)}. \tag{5.12}$$

Each $X_a, Y_b \in L$, and $z_{\alpha\beta}(a,b)$ – a sum of two distinct blocks of Z – lies in $L \cdot \overline{s}$ by Wilson's condition (ii) on Z. Hence every summand of (5.12) lies in $*_O\big(L, *_O(L, L\cdot\overline{s})\big) \subseteq L \cdot \overline{s}$ by inclusion (5.10), and so does their sum. As $\{\alpha,\beta\}$ was arbitrary, all three pair sums $W_1 + W_2$, $W_1 + W_3$, $W_2 + W_3$ lie in $L \cdot \overline{s}$.

Condition (iii): $W_1 + W_2 + W_3 \in L \cdot s$. By Lemma 5.2, $W_1 + W_2 + W_3 = X_+ *_O (Y_+ *_O Z_+)$ with the block sums $X_+, Y_+, Z_+ \in L \cdot s$ by Wilson's condition (iii) on X,Y,Z. Hence $W_1 + W_2 + W_3 \in *_O\big(L\cdot s, *_O(L \cdot s, L \cdot s)\big) \subseteq L \cdot s$ by inclusion (5.11).

The three conditions (3.3) hold for W, so $W = T_{\mathrm{Okubo}}(X,Y,Z) \in \Lambda$. Since T_{Okubo} is $\mathbb{R}$-trilinear and Λ is a lattice, $(\Lambda, +, T_{\mathrm{Okubo}})$ is a ternary order. ☐

The closure of Λ admits a second proof that uses no structural information about $*_O$: $\mathbb{R}$-trilinearity of T_{Okubo} alone reduces it to a finite exhaustive check.

Lemma 5.8 (Trilinearity reduction)**.** *Let $T : (\mathbb{R}^{24})^3 \to \mathbb{R}^{24}$ be $\mathbb{R}$-trilinear, and let $\Lambda \subset \mathbb{R}^{24}$ be a lattice of rank 24 with $\mathbb{Z}$-basis $\{b_1, \ldots, b_{24}\}$. Then $T(\Lambda,\Lambda,\Lambda) \subseteq \Lambda$ if and only if $T(b_i, b_j, b_k) \in \Lambda$ for all $i,j,k \in \{1,\ldots,24\}$.*

Proof. The forward implication is immediate, as $b_i, b_j, b_k \in \Lambda$. Conversely, any $X,Y,Z \in \Lambda$ are integer combinations $X = \sum_i x_i b_i$, $Y = \sum_j y_j b_j$, $Z = \sum_k z_k b_k$ with $x_i, y_j, z_k \in \mathbb{Z}$. By trilinearity,

$$T(X,Y,Z) = \sum_{i,j,k} x_i\, y_j\, z_k\; T(b_i, b_j, b_k), \tag{5.13}$$

an integer combination of the vectors $T(b_i,b_j,b_k) \in \Lambda$. Since Λ is a $\mathbb{Z}$-module, $T(X,Y,Z) \in \Lambda$. ☐

Remark 5.9 (Independent finite verification). Theorem 5.7 also follows directly from Lemma 5.8: since T_{Okubo} is $\mathbb{R}$-trilinear (Definition 3.8) and Λ has rank 24, the inclusion $T_{\mathrm{Okubo}}(\Lambda,\Lambda,\Lambda) \subseteq \Lambda$ is equivalent to the finite statement

$$T_{\mathrm{Okubo}}(b_i, b_j, b_k) \in \Lambda \quad \text{for all } i,j,k \in \{1,\ldots,24\}, \tag{5.14}$$

where $\{b_1, \dots, b_{24}\}$ is any fixed $\mathbb{Z}$-basis of Λ. This is a check of $24^3 = 13{,}824$ basis triples; it has been carried out exhaustively in GAP with exact rational arithmetic – a $\mathbb{Z}$-basis of Λ extracted from the 196,560 minimal vectors via `BaseIntMat`, each product $T_{\mathrm{Okubo}}(b_i, b_j, b_k)$ formed from the explicit definitions of $\cdot_\sigma$, τ, $*_O$, and the $\mathbb{Z}_3$ routing, and each tested against Wilson's criterion (3.3). All 13,824 triples pass, with zero failures [repo]. This is a *complete* proof, not a sample. It is logically independent of the structural proof above: that proof explains *why* closure holds – each Wilson condition is carried by a clean Okubo coset inclusion – while the finite reduction verifies *that* it holds by exhausting a $\mathbb{Z}$-basis.

6 Trilinear-irreducibility

The ternary order of Theorem 5.7 is not a binary order in disguise. Closure on Λ alone does not make a ternary product noteworthy: the same $\mathbb{Z}_3$ template, with the σ-twisted octonion product $\cdot_\sigma$ per block in place of $*_O$, yields ternary products T_{I}, T_{II} – routed from the 3-fold products $(a \cdot_\sigma \bar{b}) \cdot_\sigma c$ and $a \cdot_\sigma (\bar{b} \cdot_\sigma c)$ – that close on Λ *reducibly*: each is an iterate, with octonion conjugation, of the binary triple-octonion order $\star$ of the companion paper [Koeplinger2026_TripleO], and a routine finite check confirms that its closure on Λ factors through $\star$. This section shows that T_{Okubo}, built from $*_O$, admits no such reduction; we make the statement precise and prove it.

Definition 6.1 (Binary block-routed σ-twisted Okubo product)**.** Define $\circledast : \mathbb{O}^3 \times \mathbb{O}^3 \to \mathbb{O}^3$ blockwise by

$$(X \circledast Y)_\gamma \;:=\; \sum_{\substack{(\alpha,\beta)\in\{1,2,3\}^2 \\ \alpha+\beta+\gamma\equiv 0 \pmod 3}} X_\alpha *_O Y_\beta, \qquad \gamma \in \{1,2,3\}. \tag{6.1}$$

This is the natural binary block-product induced by $*_O$ together with the $\mathbb{Z}_3$ sum-zero routing on the nine cells $(\alpha, \beta) \to \gamma$. Defining $\star$ analogously with $\cdot_\sigma$ in place of $*_O$ recovers the binary triple-octonion order $(\mathbb{O}^3, \star)$ on Λ of the companion paper [Koeplinger2026_TripleO].

Lemma 6.2 (Cell-level expression of T_{Okubo})**.** *Let $\varphi : \mathbb{O}^3 \to \mathbb{O}^3$ be the block-transposition $\varphi(W_1, W_2, W_3) := (W_2, W_1, W_3)$. Then for all $X, Y, Z \in \mathbb{O}^3$,*

$$T_{\mathrm{Okubo}}(X, Y, Z) \;=\; X \circledast \big(\varphi(Y) \circledast \varphi(Z)\big). \tag{6.2}$$

Proof. Let $\psi := (1,2)$ underlie φ (so $\varphi(W)_\alpha = W_{\psi(\alpha)}$); note $\psi(j) \equiv -j \pmod 3$ for $j \in \{1,2,3\}$. For fixed γ, expand the right-hand side:

$$\big(X \circledast (\varphi(Y) \circledast \varphi(Z))\big)_\gamma = \sum_{\alpha+\beta'+\gamma\equiv 0} X_\alpha *_O \big(\varphi(Y) \circledast \varphi(Z)\big)_{\beta'} \tag{6.3}$$

$$= \sum_{\alpha+\beta'+\gamma\equiv 0} \; \sum_{\beta''+\rho''+\beta'\equiv 0} X_\alpha *_O \big(Y_{\psi(\beta'')} *_O Z_{\psi(\rho'')}\big). \tag{6.4}$$

Substitute $\beta := \psi(\beta'')$, $\rho := \psi(\rho'')$ (a bijection of $\{1,2,3\}^2$). The inner constraint becomes $\psi(\beta) + \psi(\rho) + \beta' \equiv 0$, i.e. $\beta' \equiv \beta + \rho \pmod 3$; substituting into the outer constraint $\alpha + \beta' + \gamma \equiv 0$ yields

$$\alpha + \beta + \rho + \gamma \equiv 0 \pmod 3, \tag{6.5}$$

the cell condition of $\mathcal{C}_\gamma$. The double sum in (6.4) therefore collapses to $\sum_{(\alpha,\beta,\rho)\in\mathcal{C}_\gamma} X_\alpha *_O (Y_\beta *_O Z_\rho) = T_{\mathrm{Okubo}}(X, Y, Z)_\gamma$. $\square$

Theorem 6.3 (Trilinear-irreducibility)**.** *The pair* $(\mathbb{O}^3, \circledast)$ *is not closed on* Λ*: there exist* $X, Y \in \Lambda$ *with* $X \circledast Y \notin \Lambda$*. Consequently, although* T_{Okubo} *admits the cell-level binary expression of Lemma 6.2, the closure of* Λ *under* T_{Okubo} *does not factor through any binary order on* Λ *derived from* $*_O$*: the ternary order* $(\Lambda, +, T_{\mathrm{Okubo}})$ *is genuinely trilinear.*

Proof. We track Wilson's cosets through $\circledast$. By the Petersson factorization $a *_O b = \tau(\overline{a}) \cdot_\sigma \tau^2(\overline{b})$, a pair $(a, b) \in (L \cdot \overline{s})^2$ maps under $\overline{\cdot}$ to $(s \cdot \overline{L}, s \cdot \overline{L})$ (Lemma 5.3) and under τ to itself (since $\tau(s) = s$, $\tau(\overline{L}) = \overline{L}$). Hence $*_O$ sends $(L \cdot \overline{s}) \times (L \cdot \overline{s})$ into $(s \cdot \overline{L}) \cdot_\sigma (s \cdot \overline{L})$, which is in general the *left*-coset $s \cdot \overline{L}$, *not* the right-coset $L \cdot \overline{s}$ that Wilson's condition (ii) requires. The two cosets coincide as subsets of $\mathbb{O}$ only on a proper sublocus, determined by left/right associativity defects in $\mathbb{O}$ together with the parity exchange $L \leftrightarrow \overline{L}$ on the half-integer part.

Consequently, on a given pair $(X, Y) \in \Lambda^2$ either Wilson condition (ii) or condition (iii) can fail on $X \circledast Y$; both are admissible witnesses to non-closure, the failing condition being determined by the coset structure of the input pair. An explicit witness is exhibited in Remark 6.4 below. Since $\circledast$ fails to close on Λ, the closure of Λ under T_{Okubo} – which by Lemma 6.2 runs through an inner $\circledast$-step that already leaves Λ – cannot factor through a binary order on Λ built from $*_O$. The closure is recovered only in the outer step, by a genuinely trilinear cancellation. ☐

Remark 6.4 (Status of Theorem 6.3)*.* Theorem 6.3 is proved structurally. The proof above identifies the obstruction to closure – the τ-twist sends Wilson's right-coset to the left-coset – and a single explicit pair $(X, Y) \in \Lambda^2$ with $X \circledast Y \notin \Lambda$ then suffices to establish non-closure. Writing a vector of $\mathbb{O}^3 = \mathbb{R}^{24}$ as the 3×8 array whose rows are the coordinate 8-tuples of its three octonion blocks in the basis $(e_0, \dots, e_7)$, one such witness is the pair of minimal vectors

$$X = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad Y = \tfrac{1}{2} \begin{pmatrix} -3 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 \end{pmatrix},$$

both in $\mathrm{Min}(\Lambda)$ (each of squared norm 8). Their block-routed product $W = X \circledast Y$ is

$$W = \begin{pmatrix} 1 & -3 & -3 & -1 & -1 & -3 & 1 & 1 \\ -1 & 1 & -1 & -1 & -1 & -3 & -1 & -1 \\ 3 & -1 & -1 & 1 & 1 & -1 & -1 & -1 \end{pmatrix}.$$

Each block W_γ lies in L (condition (i) holds) and each pair-sum $W_\alpha + W_\beta$ lies in $L \cdot \overline{s}$ (condition (ii) holds), but the total $W_1 + W_2 + W_3 = (3, -3, -5, -1, -1, -7, -1, -1)$, though an element of L, does *not* lie in $L \cdot s$: condition (iii) fails, so $W \notin \Lambda$. This one counterexample completes the proof. Computation [repo] corroborates, redundantly, that the failure is *generic* rather than exceptional: on 2,000,000 uniformly random pairs $(X, Y) \in \mathrm{Min}(\Lambda)^2$ the product $X \circledast Y$ lies in Λ in 1,003,114 cases (50.16%), failing on the remaining 49.84% (binomial standard error 0.04 percentage points). That statistic is not part of the proof – one witness is enough – but it shows the non-closure is not a rare edge case.

Remark 6.5 (Why $\star$ does close)*.* By contrast the binary product $\star$ *is* a closed order on Λ [Koeplinger2026_TripleO], because the σ-twisted product $\cdot_\sigma$ – without the τ-twist – preserves Wilson's right-coset structure $L \cdot \overline{s}$ directly [Koeplinger2026_TripleO]. The structural distinction is exactly the τ-twist built into $*_O$ (3.4), which sends the right-coset to the left-coset and so breaks $\circledast$'s naive Wilson-equivariance. This is what makes T_{Okubo} trilinear-irreducible while $\star$ is a binary order.

Remark 6.6 (Brown–Gray context). By [BrownGray1967, Theorem 3.1], no strict 3-ary cross-product – no fully-trilinear composition algebra – exists on $\mathbb{R}^{24}$. The $\mathbb{Z}_3$-cyclic per-block construction of T_{Okubo} is therefore the maximal-symmetry weakening admissible in dimension 24: it carries the composition-algebra structure of the σ-twisted Okubo product $(\mathbb{O}, *_O)$ forward onto Λ through Wilson's three-coset embedding $\Lambda \hookrightarrow L^3$, and Theorem 6.3 shows the resulting ternary order is not reducible to the binary case.

7 Discussion

The Leech lattice carries at least two distinct octonionic algebraic structures: the binary triple-octonion order of [Koeplinger2026_TripleO], and the ternary order $(\Lambda, +, T_{\mathrm{Okubo}})$ of Theorem 5.7. By Theorem 6.3 the second is not a reassembly of the first, nor of any binary order derived from the σ-twisted Okubo product: it is genuinely trilinear. The same $\mathbb{Z}_3$ template, with the σ-twisted octonion product in the per-block slot, instead yields ternary products that close on Λ reducibly (Section 6); the irreducibility of T_{Okubo} is what sets it apart.

The predecessor paper's outlook [Koeplinger2026_TripleO, §8.1] identifies ternary composition algebras – and Okubo's order-3 Petersson construction in particular – as a natural setting in which to look for a ternary structure on Λ compatible with the $\mathbb{Z}_3$-block symmetry. The present paper carries out that direction: T_{Okubo} is built per block from precisely that Petersson construction (with the σ-twisted product in place of the standard one) and assembled by the same $\mathbb{Z}_3$ routing that the binary order $\star$ uses. Theorem 6.3 adds that T_{Okubo} is not a trace-like reduction of any binary order derived from $*_O$ – the ternary structure of $(\Lambda, +, T_{\mathrm{Okubo}})$ is essential, not a re-packaging of a binary one. The same outlook [Koeplinger2026_TripleO, §8.2] also points to programs that model fundamental physics on normed-division-algebra, triality, or symmetric-composition-algebra structures [Koeplinger2023, FureyHughes2025_TrioOfTrialities, MarraniCorradettiZucconi2025] as one setting in which a non-classical product on $\mathbb{R}^{24}$ may be of interest.

Closure of Λ under T_{Okubo} (Theorem 5.7) is proved two independent ways: a structural proof, which shows *why* Λ closes – each of Wilson's three coset conditions is carried through T_{Okubo} by a clean coset inclusion of the σ-twisted Okubo product – and a finite reduction, which shows *that* it does, by exhausting a $\mathbb{Z}$-basis (Remark 5.9). Through the isotopy identity the structural proof reduces to coset behavior of the σ-twisted octonion product, leaving as its only computational residue three 64-entry structure-constant tables.

The maximality of $(\Lambda, +, T_{\mathrm{Okubo}})$ we leave open; one natural route to it can, however, be ruled out. The integral octonions realize E_8 as a maximal order through a *minimal-shell scaling*: octonion multiplication carries the minimal shell of E_8 into a rescaled copy of itself. No such argument is available in dimension 24.

Proposition 7.1. *No multilinear product* $T : (\mathbb{R}^{24})^m \to \mathbb{R}^{24}$*, of any arity* $m \geq 2$*, satisfies* $T\big(\mathrm{Min}(\Lambda)^m\big) \subseteq k \cdot \mathrm{Min}(\Lambda)$ *for a real* $k > 0$*.*

Proof. Since $\mathrm{Min}(\Lambda)$ is an $\mathrm{Aut}(\Lambda)$-orbit spanning $\mathbb{R}^{24}$ on which N is nonzero, the symmetric squares $v \otimes v$, $v \in \mathrm{Min}(\Lambda)$, span $\mathrm{Sym}^2(\mathbb{R}^{24})$: they span a nonzero $\mathrm{Aut}(\Lambda)$-submodule of $\mathrm{Sym}^2(\mathbb{R}^{24}) = \mathbb{R}\,N \oplus \mathbf{299}$, whose summands are irreducible and inequivalent [ConwaySloane1999], and that submodule is neither $\mathbb{R}\,N$ (the $v \otimes v$ have rank 1) nor contained in $\mathbf{299}$ ($\mathrm{tr}(v \otimes v) = N(v) \neq 0$). Hence every quadratic form constant on $\mathrm{Min}(\Lambda)$ is a scalar multiple of N. Now suppose $T\big(\mathrm{Min}(\Lambda)^m\big) \subseteq k \cdot \mathrm{Min}(\Lambda)$. Fixing all but one argument in $\mathrm{Min}(\Lambda)$, the map $X \mapsto |T(\ldots, X, \ldots)|^2$ is a quadratic form constant on $\mathrm{Min}(\Lambda)$, hence a multiple of N for *all* X; applied slot by slot, $|T(X_1, \ldots, X_m)|^2 =$

$c\,N(X_1)\cdots N(X_m)$ identically. Fixing $m-2$ arguments to unit vectors and rescaling yields a bilinear $B:\mathbb{R}^{24}\times\mathbb{R}^{24}\to\mathbb{R}^{24}$ with $|B(x,y)|^2 = N(x)\,N(y)$ – a composition of quadratic forms – which by Hurwitz's theorem [Hurwitz1898] exists only in dimensions $\{1,2,4,8\}$. Since $24\notin\{1,2,4,8\}$, no such T exists. □

Carrying the minimal shell into a rescaled copy of itself is *sufficient* for maximality but not necessary, so Proposition 7.1 closes this one route without settling the maximality question itself.

Beyond the Leech lattice itself, the construction turns up an algebra of independent interest. The per-block product $(\mathbb{O}, *_O)$ – the σ-twisted Okubo product – is a composition algebra that is *not* symmetric, hence outside Elduque's classification; its automorphism group is only $U(2)$, against the $SU(3)$ of the Okubo algebra and the G_2 of the octonions (Section 4). It is a concrete non-symmetric composition-algebra isotope, and its relation to the integral Okubo algebra [Corradetti2026] – in particular the open question of Section 4.3, whether the recurrence of E_8 and the Leech lattice across these Okubo-type constructions has a structural cause – is a natural direction for further work.

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A Data for the coset inclusions of Lemma 5.5

This appendix records the data – the σ-twisted product and explicit $\mathbb{Z}$-bases of the lattices involved – from which the three coset inclusions (5.6)–(5.8) of Lemma 5.5 can be verified directly.

The σ-twisted product. With $\sigma = (1,2)$ (Definition 3.1), the σ-twisted octonion product is given on the basis $\{e_0,\dots,e_7\}$ by the table

$\cdot_\sigma$	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-e_0$	$-e_4$	e_5	e_2	$-e_3$	e_7	$-e_6$
e_2	e_2	e_4	$-e_0$	e_7	$-e_1$	e_6	$-e_5$	$-e_3$
e_3	e_3	$-e_5$	$-e_7$	$-e_0$	e_6	e_1	$-e_4$	e_2
e_4	e_4	$-e_2$	e_1	$-e_6$	$-e_0$	e_7	e_3	$-e_5$
e_5	e_5	e_3	$-e_6$	$-e_1$	$-e_7$	$-e_0$	e_2	e_4
e_6	e_6	$-e_7$	e_5	e_4	$-e_3$	$-e_2$	$-e_0$	e_1
e_7	e_7	e_6	e_3	$-e_2$	e_5	$-e_4$	$-e_1$	$-e_0$

with $e_i \cdot_\sigma e_j$ in row i, column j, extended $\mathbb{R}$-bilinearly.

The lattices. A $\mathbb{Z}$-basis $\{b_1,\dots,b_8\}$ of the integer-octonion lattice L (Definition 3.2) is

$$b_1 = \tfrac{1}{2}(e_0+e_1+e_2+e_3+e_4+e_5+e_6+3e_7), \qquad b_{k+1} = e_k + e_7 \ \ (1\le k\le 6), \qquad b_8 = 2e_7;$$

conjugating these gives a $\mathbb{Z}$-basis of $\overline{L}$. With $s = \frac{1}{2}(-e_0+e_1+\cdots+e_7)$ and its conjugate $\overline{s}$ (Definition 3.2), the cosets occurring in Lemma 5.5 have $\mathbb{Z}$-bases $L\cdot s = \{b\cdot s\}$, $L\cdot\overline{s} = \{b\cdot\overline{s}\}$, $s\cdot\overline{L} = \{s\cdot c\}$, and $\overline{s}\cdot\overline{L} = \{\overline{s}\cdot c\}$, with b ranging over the basis of L and c over the basis of $\overline{L}$.

The verification. For each of the three inclusions (5.6)–(5.8), the 8×8 array of products $b_i \cdot_\sigma c_j$ of source-lattice basis vectors, expressed in the $\mathbb{Z}$-basis of the target lattice, was formed in exact arithmetic; all $3 \times 64 = 192$ coordinate vectors are integral, which by $\mathbb{Z}$-bilinearity establishes the three inclusions [repo].

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