Refuting the originality of M. Gogberashvili, A. Gurchumelia, “Split octonionic Dirac equation”

10 Dec 2025 (v9)

Jens Köplinger

jenskoeplinger@gmail.com

Abstract

In a recent paper (M. Gogberashvili, A. Gurchumelia, “Split octonionic Dirac equation” [GG2024]) the authors claim discovery of “novel forms of the split octonionic Dirac equation.” This research brief refutes the originality claim in a focused side-by-side demonstration of existing publication (J. Köplinger, “Dirac equation on hyperbolic octonions” [Koepl2006]), coauthored follow-on work (J. Köplinger, V. Dzhunushaliev, M. Gogberashvili, “Emergent time from non-associative quantum theory” [KDG2008]), and the contended work at hand. Disputed points are isolated and separated from the actual accomplishments. In addition, this brief provides a review section with four distinct approaches to (split-)octonions and the Dirac equation, along with pointers to their modern use.

Contents

1 Split-octonions (hyperbolic octonions)

Split-octonions $\mathbb{O}'$ are eigth-dimensional unital composition algebras [Jacob1958] that are typically understood over the real numbers. Any number $x \in \mathbb{O}'$ is uniquely represented as a vector $x = (x_0, \ldots, x_7) \in \mathbb{R}^8$ after fixing a vector space basis of $\mathbb{R}^8$. Subsequently, the algebra $\langle \mathbb{O}' \mid * \mid + \rangle$ is specified by requiring existence of a multiplicative unit $e_0 \in \mathbb{O}'$ and fixing a multiplication rule that satisfies product-composition of a quadratic form $\|\cdot\|$:

$$\|\cdot\| : \mathbb{O}' \to \mathbb{R}, \tag{1.1}$$

$$\|x\| := \left(x_0^2 + x_1^2 + x_2^2 + x_3^2\right) - \left(x_4^2 + x_5^2 + x_6^2 + x_7^2\right), \tag{1.2}$$

$$\|x * y\| = \|x\| \, \|y\| \text{ for any } x, y \in \mathbb{O}'. \tag{1.3}$$

Here, the vector space basis is chosen such that x_0 through x_3 contribute with a positive sign to the quadratic form, and x_4 through x_7 contribute with a negative sign. By convention, the 0-dimension is typically identified with the real number line, such that $e_0 := (1, 0, \ldots, 0)$ is the multiplicative unit. These choices are of course far from unique, and many equivalent representations for split-octonion multiplication exist that satisfy these constraints. We will not go into detail around possible choices, algebra isomorphisms and automorphisms ("symmetries") in this brief. For reviews see e.g. [Okubo1995, Baez2002, CS2003].

Looking at the definition of the quadratic form (1.2), the fact that the number of positive and negative signs split the eight coefficients in two groups of four gives rise to the commonly used modern terminology "split-octonions". Conversely, the isotropic form $\|\cdot\|$ contains hyperbolic planes in various two-dimensional subspaces, hence equivalent terminology "hyperbolic octonions" for the same algebra.

2 Dirac equation as direct binary split-octonion product

2.1 J. Köplinger, "Dirac equation on hyperbolic octonions" (2006)

The Dirac equation in physics is the fundamental equation of motion of a free spin-$\frac{1}{2}$ particle. It uses the particle's mass $m \in \mathbb{R}$, partial differential operators $\partial_\mu := \partial / \partial x_\mu$ (with $\mu = 0, 1, 2, 3$) in space $x_1, x_2, x_3 \in \mathbb{R}$ and time $x_0 \in \mathbb{R}$, and a complex-valued wave function that is a function of spacetime, $\Psi : \mathbb{R}^4 \to \mathbb{C}^4 \cong \mathbb{R}^8$, $\Psi = \left(\psi_0^{\mathrm{r}} + i\psi_0^{\mathrm{i}}, \ldots, \psi_3^{\mathrm{r}} + i\psi_3^{\mathrm{i}}\right)$, where ψ_μ^{r} denotes the real part of the μ-component of Ψ and ψ_μ^{i} its imaginary part. Customarily written by means of Dirac matrices γ^μ that form the basis of a Clifford algebra, the equation becomes[1]

$$\left(\sum_{\mu=0}^{3} i\gamma^\mu \partial_\mu - m \right) \Psi = 0. \tag{2.1}$$

This linear equation can be separated into eight real expressions.

Conversely, any binary split-octonion product can be separated into eight real expressions likewise, simply by writing out the component summations for each of the eight dimensions. In 2006 this author demonstrated [Koepl2006] that it is possible to write the eight expressions of the Dirac equation in symbolic form as a direct binary split-octonion product, using only two

[1] Many different conventions exist around physical constants (like $\hbar$ and c), index notation, summation, and metric tensor $\eta_{\mu\nu}$. Here we just give a simplified overview, for details see the canonical literature.

factors $\nabla\Psi = 0$, by pairwise identifying all components. Written to split-octonion basis element $b_{\mathbb{O}'} = \{1, i_1, i_2, i_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7\}$, equations (3), (4), and (5) in that publication are[2]:

$$\Psi := \left(\psi_0^{\mathrm{r}}, \psi_0^{\mathrm{i}}, \psi_1^{\mathrm{r}}, \psi_1^{\mathrm{i}}, \psi_2^{\mathrm{r}}, -\psi_2^{\mathrm{i}}, -\psi_3^{\mathrm{r}}, -\psi_3^{\mathrm{i}}\right), \tag{2.2}$$

$$\nabla := \left(-m, \partial_0, 0, 0, 0, -\partial_3, \partial_2, -\partial_1\right), \tag{2.3}$$

$$\nabla\Psi = 0. \tag{2.4}$$

2.2 J. Köplinger, V. Dzhunushaliev, M. Gogberashvili, "Emergent time from non-associative quantum theory" (2008)

In an effort to connect the initial finding into existing related context, this author was happy to collaborate for an essay contest [KDG2008]. The goal was to raise awareness of the research domain, make it more accessible and venture a thought-provoking advertisement for split-octonions in physics in general.

The essay bridged various referenced works. It also used the more modern terminology "split-octonion" and clarified where alternate terminology "hyperbolic octonions" arises from:

> "Just as the octonions, the split-octonions have a multiplicative 2-form; only this time it is not Euclidean, but hyperbolic, ... The hyperbolic 2-form of split-octonions (20) ..."

Equations (27), (28), and (29) therein refer to this author's follow-on work [Koepl2007c] that takes advantage of the simplicity of binary split-octonion product representation of the Dirac equation [Koepl2006], to introduce expansions in the form of a mixing angle $\alpha \in \mathbb{R}$, four-potential $(A_0, \ldots, A_3)$ and charge e. Labeling octonion basis elements explicitly as $b_{\mathbb{O}} = (i_0, \ldots i_7)$, these equations are:

$$\begin{aligned} \nabla_{\mathrm{A}} &:= i_1 \left(\partial_0 - i_0 e A_0\right), \\ \nabla_{\mathrm{B}} &:= i_5 \left(\partial_3 - i_0 e A_3\right) + i_6 \left(-\partial_2 + i_0 e A_2\right) + i_7 \left(\partial_1 - i_0 e A_1\right), \\ \nabla &:= \nabla_{\mathrm{A}} + e^{i_0\alpha} \nabla_{\mathrm{B}}, \end{aligned} \tag{2.5}$$

$$\begin{aligned} \Psi_{\mathrm{A}} &:= \psi_0 + i_1\psi_1 + i_2\psi_2 + i_3\psi_3, \\ \Psi_{\mathrm{B}} &:= -i_4\psi_4 + i_5\psi_5 + i_6\psi_6 + i_7\psi_7, \\ \Psi &:= \Psi_{\mathrm{A}} + e^{i_0\alpha} \Psi_{\mathrm{B}}, \end{aligned} \tag{2.6}$$

$$\left(\nabla - m\right)\Psi = 0. \tag{2.7}$$

The reader easily verifies that this reduces to this author's representation of the Dirac equation (2.4) when setting $\alpha = \frac{\pi}{2}$, the potentials $(A_0, \ldots, A_3)$ to zero, trivially relabeling the real wave function components, and pairwise identifying split-octonion basis elements $b_{\mathbb{O}'} = \{1, i_1, i_2, i_3, -i_0 i_4, \ldots - i_0 i_7\} \equiv \{1, i_1, i_2, i_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7\}$. The essay properly references this in "[7]".

[2] Here we use the corrected 2007 form of [Koepl2006], following the corrigendum in [Koepl2007c] footnote 2. For a self-contained corrigendum with acknowledgment see e.g. ResearchGate brief DOI: 10.13140/RG.2.2.22900.41602 or this author's personal publications web page [KoeplWWW]. The original subscript "hyp8" is omitted as it does not add value here.

2.3 M. Gogberashvili, A. Gurchumelia, "Split octonionic Dirac equation" (2024)

Split-octonions in [GG2024] are written to basis elements $b_{\mathbb{O}'} = \{1, j_1, j_2, j_3, I, J_1, J_2, J_3\}$. A wave function is initially defined in equation (4) as

$$\psi = \begin{pmatrix} \psi_4 + i\psi_7 \\ -\psi_6 + i\psi_5 \\ \psi_3 + i\psi_0 \\ \psi_1 + i\psi_2 \end{pmatrix} \tag{2.8}$$

and later (14) embedded into $\mathbb{O}'$ as

$$\psi = \psi_0 + I\psi_4 + \sum_{n=1}^{3} \left(j_n\psi_n + J_n\psi_{4+n} \right). \tag{2.9}$$

This associates one real component of the wave function with one split-octonion basis, and the function is then assumed "constant in variables x_0, x_5, x_6 and x_7", i.e., $\psi : \mathbb{R}^4 \to \mathbb{R}^8$ just as in 2.2 above.

A differential operator $\mathscr{D}$ (15) acts on the variables x_1, x_2, x_3 and $x_4 \equiv t$,

$$\mathscr{D} = I\partial_t - \sum_{n=1}^{3} j_n\partial_n, \tag{2.10}$$

such that the split-octonionic Dirac equation can be represented as (16)

$$\left(\mathscr{D} - J_3 m \right) \psi = 0, \tag{2.11}$$

which the authors then show to be "component-wise equivalent to the standard Dirac system" with the help of a computer program. The structure of (2.11) is that of a direct binary product of split-octonions, and is therefore of identical nature as (2.4), the only difference being relabeling of coefficients and vector space basis element embeddings.

For the sake of clarity, we demonstrate this explicitly. We form a new operator $\mathscr{D}'$ by left-multiplying $\mathscr{D}$ (2.10) with J_3, using the multiplication rules stated in [GG2024]:

$$\mathscr{D}' := J_3\mathscr{D} = j_3\partial_t + J_2\partial_1 - J_1\partial_2 - I\partial_3. \tag{2.12}$$

Right-multiplying ψ (2.8) with J_3 yields:

$$\psi' := \psi J_3 \tag{2.13}$$
$$= J_3\psi_0 - j_3\psi_4 + J_2\psi_1 - j_2\psi_5 - J_1\psi_2 + j_1\psi_6 - I\psi_3 + \psi_7. \tag{2.14}$$

Because split-octonions satisfy the Moufang property $(zx)(yz) = (z(xy))z$, the expression

$$\left(\mathscr{D}' - m \right) \psi' = \left(J_3\mathscr{D} - J_3 \left(J_3 m \right) \right) \left(\psi J_3 \right) = \left(J_3 \left(\left(\mathscr{D} - J_3 m \right) \psi \right) \right) J_3 \tag{2.15}$$

is a structure-preserving rotation[3] of $\left(\mathscr{D} - J_3 m \right) \psi$ using J_3, here specifically a nonassociative generalization of group conjugation, aba^{-1}.

[3] For details on general rotations in the space of split-octonions see e.g. [Gogb2008].

Writing $\mathscr{D}'$ and ψ' as split-octonions, with basis elements embedded into $\mathbb{R}^8$ reordered to $b'_{\mathbb{O}'} = \{1, j_3, j_2, j_1, -J_3, I, -J_1, -J_2\}$,

$$\psi' = (\psi_7, \psi_4, -\psi_5, \psi_6, -\psi_0, -\psi_3, \psi_2, -\psi_1)\,, \tag{2.16}$$
$$\mathscr{D}' - m = (-m, \partial_t, 0, 0, 0, -\partial_3, \partial_2, -\partial_1)\,. \tag{2.17}$$

this becomes identical to (2.2) and (2.3) after trivial relabeling of the real components of the wave function ψ'. Pairwise identifying $b'_{\mathbb{O}'} = \{1, j_3, j_2, j_1, -J_3, I, -J_1, -J_2\} \equiv \{1, j'_1, j'_2, j'_3, I', J'_1, J'_2, J'_3\} \equiv \{1, i_1, i_2, i_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7\}$ allows to confirm that $b'_{\mathbb{O}'}$ is indeed a split-octonion basis.

The following claims in [GG2024] are therefore untenable:

- "The novel form of the split octonionic Dirac equation ..." has in fact previously been described by this author [Koepl2006], namely, as a *2-factor representation* that is a direct split-octonion product.
- "In conclusion, this paper has successfully derived the standard Dirac equation within the framework of split octonions" is instead an independent verification of this author's description.
- "Methods of calculations and obtained split octonionic Dirac equation ... are slightly different from earlier findings in [8] *[ann.:* [Gogb2006b]*]*. The main difference is the appearance of split octonionic imaginary unit J_3 in these expressions" is both incorrect and misleading. It is incorrect because J_3 generates a structure-preserving rotation, shown in equation (2.15) above. This rotation amounts to an algebra automorphism, and therefore cannot be of substantial difference in general[4]. It is also misleading since the vague expression "slightly different" references work [Gogb2006b] that uses a structurally different 3-*factor representation* of the Dirac equation. That representation is made from a sum of factors using three split-octonion basis elements each, and is in fact leaning on prior work by different authors [DLAK1996a, DLAK1996b].

In conclusion, the authors of [GG2024] accomplished an independent verification of the finding in [Koepl2006] and demonstrated their methods in the form of a computer program. They also tied in existing other work by the authors around algebraic modeling of a Lagrangian.

3 Four approaches to (split-)octonions and the Dirac equation

Without claiming completeness, this section provides select pointers to other existing work on (split-)octonions around the Dirac equation. This author is aware of four conceptually different approaches: The *2-factor*, *3-factor*, and *projection* representations using octonion basis elements natively for modeling some spacetime basis; and separately, *conventional* Dirac algebra using octonionic *spinors*.

This brief focused on the *2-factor representation* that expresses the Dirac equation as a direct split-octonion product [Koepl2006], as detailed above. We point out that this work, in turn, followed up on an obscure mention in [Muses1980]:

> "... Dirac's equation ... although a simpler version of the equation using only 16-dimensional M-algebra *[ann.: complex octonions]* is possible ..."

[4] The authors themselves investigate this e.g. in [GG2019].

No proof was given. An octonionic 2-factor representation in the complex octonions was indeed shown in [CSS2000] with the help of an 8×8 matrix construction. That construction also includes other physical observables in the form of additional mass-like terms.

The *3-factor representation* after S. De Leo and K. Abdel-Khalek [DLAK1996a, DLAK1996b] expresses the Dirac equation as a sum of terms that generally consist of three split-octonion basis elements. It was adopted by M. Gogberashvili [Gogb2006a, Gogb2006b], and later followed up with an elegant mathematical treatment by R. Beradze and T. Shengelia [BS2016].

A different program has its roots in work by A. Sudbery (with K. W. Chung) [Sudb1987, CS1987], who use octonion basis elements in a 2×2 matrix setting in a treatment primarily in terms of Lie algebra. Starting with a 10-dimensional configuration space [SM1994], T. Dray and C. A. Manogue [MD1999, DM2000] use an algebraic *projection* to retrieve faithful representations of the (associative) Dirac equation (2.1), in a way that resembles three particle families together with a sterile neutrino.

The *2-factor*, *3-factor*, and *projection* representations express the Dirac equation to split-octonion basis elements natively, which introduces nonassociativity in a way that requires clarification when connecting to canonical formulation of physical law. In contrast, octonions can also be used to model *conventional* Dirac algebra acting on certain *spinors* in an elegant algebraic way. This appeared as early as the papers of I. Bengtsson and M. Cederwall [BC1988], R. Foot and G. C. Joshi [FJ1988], and S. Marques-Bonham [MarBon1991]. Works by G. M. Dixon [Dixon1994], J. Schray and C. Manogue [SM1994], J. C. Baez and J. Huerta [BH2010], and N. Furey [Furey2012, Furey2016, Furey2025] are just a few prominent modern examples in this investigation towards driving physical laws from the nature of numbers.

There appears to be considerable confusion in the literature about the proper context of one's work when it comes to octonions and the Dirac equation. To give an example, the projection approach from Dray and Manogue [MD1999, DM2000] appears to have been picked up recently by J. C. Vélez Quiñones in [VQ2022]; however, that author quotes De Leo and Abdel-Khalek [DLAK1996b] instead, which does not appear to be used in that paper. Ironically, in a recent work coauthored by Manogue and Dray [MDW2022] the authors cite their projection approach [MD1999, DM2000], even though no such projection is used or needed; instead, their construction of Lie algebras from one-sided octonionic multiplication would lend itself towards this author's direct product [Koepl2006] when looking for spacetime representations in not necessarily associative subspaces. Confusion aside, references to the Dirac equation in these examples are made "in passing" and not substantial for the respective findings.

Finally, we would like to point to recent mathematical methods for native split-octonionic analysis [KLL2025, LZ2025], which could make many of the above symbolic expressions on nonassociative algebra analytically tractable. This includes a large body of contemporary work [CTÖD2008, Weng2009, TKD2012, Demir2013, DTK2013, TKD2014, CSN2015, MBCM2020, Koepl2023a, Koepl2023b, Lasen2023], to mention just a few, with apologies for other original approaches that were inadvertently missed. With almost 100 years of knowledge of both the Dirac equation and split-octonions, it is highly likely that more preexisting work has yet to be properly acknowledged.

Acknowledgments

While I'm writing the refutation solely in my own capacity, I am grateful for all the impulses, pointers, inspiration, help, support, and references in the early days that lead to the original 2006 paper. It is an honor to have written and published in the context of those who helped me, to have

acknowledged you in the relevant publications and talks over the years [KoeplWWW], and to now place the original work into a wider context in the form of a review section. Also, I am thankful to several mentors for their guidance on how to deal with the situation at hand, to protect my academic claim while at the same time improving on the literature, and to lay the groundwork for correcting contemporary popular internet search and GenAI engines referring to the disputed work at hand when being asked for a split-octonionic version of the Dirac equation.

References

[Baez2002] J. C. Baez, The Octonions, *Bull. Amer. Math. Soc.* **39** (2002), 145-205; *arXiv*:math/0105155v4 [math.RA].

[BC1988] I. Bengtsson, M. Cederwall, Particles, twistors and the division algebras, *Nulc. Phys. B* **302** (1988), 81-103.

[BH2010] J. C. Baez, J. Huerta, "Division Algebras and Supersymmetry I" in *Superstrings, Geometry, Topology, and C*-algebras*, eds. R. Doran, G. Friedman and J. Rosenberg; *Proc. Symp. Pure Math.* **81** (2010) 65-80; *arXiv*:0909.0551 [hep-th].

[BS2016] R. Beradze, T. Shengelia, Dirac and Maxwell equations in Split Octonions, *arXiv* (2016); *arXiv*:1610.06418 [physics.gen-ph].

[CSN2015] B. C. Chanyal, V. K. Sharma, O. P. S. Negi, Octonionic Gravi-Electromagnetism and Dark Matter, *Int. J. Theor. Phys.* **54** (2015), 3516–3532.

[CS1987] K. W. Chung, A. Sudbery, Octonions and the Lorentz and Conformal Groups of Ten-dimensional Space-time, *Phys. Lett. B* **198** (1987), 161-164.

[CS2003] J. H. Conway, D. Smith, On Quaternions and Octonions (2003), CRC Press, Taylor & Francis Group.

[CSS2000] F. Colombo, I. Sabadini, D. C. Struppa, Dirac Equation in the Octonionic Algebra, in *Analysis, Geometry, Number Theory: The Mathematics of Leon Ehrenpreis*, eds. E. L. Grinberg, S. Berhanu, M. Knopp, G. Mendoza, E. T. Quinto; *Contemporary Math.* **251** (2000), 117–134.

[CTÖD2008] N. Candemir, M. Tanışlı, K. Özdaş, S. Demir, Hyperbolic Octonionic Proca-Maxwell Equations, *Z. Naturforsch.* **63a** (2008), 15-18.

[Demir2013] S. Demir, Hyperbolic Octonion Formulation of Gravitational Field Equations, *Int. J. Theor. Phys.* **52** (2013), 105-116.

[Dixon1994] G. M. Dixon, Division Algebras: Octonions, Quaternions, Complex Numbers and the Algebraic Design of Physics (1994), Springer US.

[DLAK1996a] S. De Leo, K. Abdel-Khalek, Octonionic Quantum Mechanics and Complex Geometry, *Prog. Theor. Phys.* **96** (1996) 823-832; *arXiv*:hep-th/9609032.

[DLAK1996b] S. De Leo, K. Abdel-Khalek, Octonionic Dirac Equation, *Prog. Theor. Phys.* **96** (1996) 833-846; *arXiv*:hep-th/9609033.

[DM2000] T. Dray, C. A. Manogue, Quaternionic Spin, in *Clifford Algebras and their Applications in Mathematical Physics*, eds. R. Abłamowicz, B. Fauser (Birkhäuser, Boston, 2000); *arXiv*:hep-th/9910010.

[DTK2013] S. Demir, M Tanışlı, M. E. Kansu, Generalized Hyperbolic Octonion Formulation for the Fields of Massive Dyons and Gravito-Dyons, *Int. J. Theor. Phys.* **52** (2013), 3696-3711.

[FJ1988] R. Foot, G. C. Joshi, On a certain supersymmetric identity and the division alebras, *Mod. Phys. Lett. A* **3** (1988), 999-1004.

[Furey2012] C. Furey, Unified Theory of Ideals, *Phys. Rev. D* **86** (2012), 025024; *arXiv*:1002.1497 [hep-th].

[Furey2016] C. Furey, "Standard model physics from an algebra?", *Ph.D. Thesis* (2016); *arXiv*:1611.09182 [hep-th].

[Furey2025] N. Furey, A Superalgebra Within: Representations of Lightest Standard Model Particles Form a $\mathbb{Z}_2^5$-Graded Algebra, *Ann. Phys.* (2025) e00229, DOI: 10.1002/andp.202500229; *arXiv*:2505.07923 [hep-ph].

[GG2019] M. Gogberashvili, A. Gurchumelia, Geometry of the Non-Compact G(2), J. Geom. Phys. **144** (2019), 308-313; *arXiv*:1903.04888 [physics.gen-ph].

[GG2024] M. Gogberashvili, A. Gurchumelia, Split octonionic Dirac equation, *Int. J. Geom. Meth. Mod. Phys.* **21** (2024), 2450214; *arXiv*:2409.13736 [physics.gen-ph].

[Gogb2006a] M. Gogberashvili, Octonionic Electrodynamics, *J. Phys. A: Math. Gen.* **39** (2006) 7099-7104; *arXiv*:hep-th/0512258.

[Gogb2006b] M. Gogberashvili, Octonionic Version of Dirac Equations, *Int. J. Mod. Phys. A* **21** (2006), 3513-3524; *arXiv*:hep-th/0505101.

[Gogb2008] M. Gogberashvili, Rotations in the space of the split octonions, Adv. Math. Phys. **2009** (2009), 483079; *arXiv*:0808.2496 [math-ph].

[Jacob1958] N. Jacobson, Composition algebras and their automorphisms, *Rendiconti del Circolo Matematico di Palermo* **7** (1958), 55-80.

[KDG2008] J. Köplinger, V. Dzhunushaliev, M. Gogberashvili, Emergent Time from Non-Associative Quantum Theory, *FQXi essay* (2008), https://fqxi.org/community/forum/topic/307; *ResearchGate* DOI: 10.13140/RG.2.2.10115.99361.

[KLL2025] R. S. Kraußhar, A. Legatiuk, D. Legatiuk, Discrete octonionic analysis: a unified approach to the split-octonionic and classical settings, *arXiv* (2025); *arXiv*:2502.02227 [math.CV].

[KoeplWWW] J. Köplinger, personal web site (publications, personal versions, talks), http://www.jenskoeplinger.com/P (retrieved 1 Nov 2025).

[Koepl2006] J. Köplinger, Dirac equation on hyperbolic octonions, *Appl. Math. Computation*, **182** (2006), 443-446; corrigendum in [Koepl2007c] footnote 2, as well as *ResearchGate* DOI: 10.13140/RG.2.2.22900.41602.

[Koepl2007c] J. Köplinger, Gravity and electromagnetism on conic sedenions, *Appl. Math. Computation*, **188** (2007), 948-953.

[Koepl2023a] J. Köplinger, Towards autotopies of normed composition algebras in algebraic Quantum Field Theory, *ResearchGate* (2023), DOI: 10.13140/RG.2.2.19003.39209.

[Koepl2023b] J. Köplinger, Phenomenology from Dirac equation with Euclidean-Minkowskian "gravity phase", *Int. J. Theor. Phys.* **62** (2023), 35; *arXiv*:2302.10748 [physics.gen-ph].

[Lasen2023] A. Lasenby, Some recent results for SU(3) and Octonions within the Geometric Algebra approach to the fundamental forces of nature, *Math. Methods Appl. Sci.* **47** (2023), 1471-1491; *arXiv*:2202.0673 [physics.gen-ph].

[LZ2025] A. Lopatin, A. N. Zubkov, On linear equations over split octonions, arXiv (2025); *arXiv*:2411.08500 [math.RA].

[MarBon1991] S. Marques-Bonham, The Dirac equation in a non-Riemannian manifold III: An analysis using the algebra of quaternions and octonions, *J. Math. Phys.* **32** (1991), 1383-1394.

[MBCM2020] S. Marques-Bonham, B. C. Chanyal, R. Matzner, Yang-Mills-like field theories built on division quaternion and octonion algebras, *Eur. Phys. J. Plus* **135** (2020), 608.

[MD1999] C. A. Manogue, T. Dray, Dimensional Reduction, *Mod. Phys. Lett.* **A14** (1999) 99-103; *arXiv*:hep-th/9807044.

[MDW2022] C. A. Manogue, T. Dray, R. A. Wilson, Octions: An E_8 description of the Standard Model, *J. Math. Phys.* **63** (2022), 081703; *arXiv*:2204.05310 [hep-ph].

[Muses1980] C. Musès, Hypernumbers and quantum field theory with a summary of physically applicable hypernumber arithmetics and their geometries, *Appl. Math. Comput.* **6** (1980), 63-94.

[Okubo1995] S. Okubo, Introduction to Octonion and Other Non-Associative Algebras in Physics (1995), Cambridge Univ. Press.

[SM1994] J. Schray, C. Manogue, Octonionic representations of Clifford algebras and triality, *Found. Phys.* **26** (1996) 17-70; *arXiv*:hep-th/9407179.

[Sudb1987] A. Sudbery, Division algebras, (pseudo)orthogonal groups and spinors, *J. Phys. A* **17** (1987), 939-955.

[TKD2012] M. Tanışlı, M. E. Kansu, S. Demir, A new approach to Lorentz invariance in electromagnetism with hyperbolic octonions, *Eur. Phys. J. Plus* **127** (2012), 69.

[TKD2014] M. Tanışlı, M. E. Kansu, S. Demir, Reformulation of electromagnetic and gravito-electromagnetic equations for Lorentz system with octonion algebra, *Gen. Relativ. Gravit.* **46** (2014), 1739.

[VQ2022] J. C. Vélez Quiñones, A unified field theory from a complexified quaternion-octonion Dirac equation, *arXiv* (2022); *arXiv*:2205.06657 [physics.gen-ph].

[Weng2009] Z. Weng, Field equations of electromagnetic and gravitational fields, *arXiv* (2009); *arXiv*:0709.2486 [physics.gen-ph].