

# Refuting the originality of M. Gogberashvili, A. Gurchumelia, “Split octonionic Dirac equation”

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*Jens Köplinger*

jenskoeplinger@gmail.com

## Abstract

In a recent paper (M. Gogberashvili, A. Gurchumelia, “Split octonionic Dirac equation” [GG2024]) the authors claim discovery of “novel forms of the split octonionic Dirac equation.” This research brief refutes the originality claim in a focused side-by-side demonstration of existing publication (J. Köplinger, “Dirac equation on hyperbolic octonions” [Koepl2006]), coauthored follow-on work (J. Köplinger, V. Dzhunushaliev, M. Gogberashvili, “Emergent time from non-associative quantum theory” [KDG2008]), and the contended work at hand. Disputed points are isolated and separated from the actual accomplishments. In addition, this brief provides a review section with four distinct approaches to (split-)octonions and the Dirac equation, along with pointers to their modern use.

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## 1 Split-octonions (hyperbolic octonions)

Split-octonions  $\mathbb{O}'$  are eighth-dimensional unital composition algebras [Jacob1958] that are typically understood over the real numbers. Any number  $x \in \mathbb{O}'$  is uniquely represented as a vector  $x = (x_0, \dots, x_7) \in \mathbb{R}^8$  after fixing a vector space basis of  $\mathbb{R}^8$ . Subsequently, the algebra  $(\mathbb{O}' \mid * \mid +)$  is specified by requiring existence of a multiplicative unit  $e_0 \in \mathbb{O}'$  and fixing a multiplication rule that satisfies product-composition of a quadratic form  $\|\cdot\|$ :

$$\|\cdot\| : \mathbb{O}' \rightarrow \mathbb{R}, \quad (1.1)$$

$$\|x\| := (x_0^2 + x_1^2 + x_2^2 + x_3^2) - (x_4^2 + x_5^2 + x_6^2 + x_7^2), \quad (1.2)$$

$$\|x * y\| = \|x\| \|y\| \text{ for any } x, y \in \mathbb{O}'. \quad (1.3)$$

Here, the vector space basis is chosen such that  $x_0$  through  $x_3$  contribute with a positive sign to the quadratic form, and  $x_4$  through  $x_7$  contribute with a negative sign. By convention, the 0-dimension is typically identified with the real number line, such that  $e_0 := (1, 0, \dots, 0)$  is the multiplicative unit. These choices are of course far from unique, and many equivalent representations for split-octonion multiplication exist that satisfy these constraints. We will not go into detail around possible choices, algebra isomorphisms and automorphisms (“symmetries”) in this brief. For reviews see e.g. [Okubo1995, Baez2002, CS2003].

Looking at the definition of the quadratic form (1.2), the fact that the number of positive and negative signs split the eight coefficients in two groups of four gives rise to the commonly used modern terminology “split-octonions”. Conversely, the isotropic form  $\|\cdot\|$  contains hyperbolic planes in various two-dimensional subspaces, hence equivalent terminology “hyperbolic octonions” for the same algebra.

## 2 Dirac equation as direct binary split-octonion product

### 2.1 J. Köpflinger, “Dirac equation on hyperbolic octonions” (2006)

The Dirac equation in physics is the fundamental equation of motion of a free spin- $\frac{1}{2}$  particle. It uses the particle’s mass  $m \in \mathbb{R}$ , partial differential operators  $\partial_\mu := \partial/\partial x_\mu$  (with  $\mu = 0, 1, 2, 3$ ) in space  $x_1, x_2, x_3 \in \mathbb{R}$  and time  $x_0 \in \mathbb{R}$ , and a complex-valued wave function that is a function of spacetime,  $\Psi : \mathbb{R}^4 \rightarrow \mathbb{C}^4 \cong \mathbb{R}^8$ ,  $\Psi = (\psi_0^r + i\psi_0^i, \dots, \psi_3^r + i\psi_3^i)$ , where  $\psi_\mu^r$  denotes the real part of the  $\mu$ -component of  $\Psi$  and  $\psi_\mu^i$  its imaginary part. Customarily written by means of Dirac matrices  $\gamma^\mu$  that form the basis of a Clifford algebra, the equation becomes<sup>1</sup>

$$\left( \sum_{\mu=0}^3 i\gamma^\mu \partial_\mu - m \right) \Psi = 0. \quad (2.1)$$

This linear equation can be separated into eight real expressions.

Conversely, any binary split-octonion product can be separated into eight real expressions likewise, simply by writing out the component summations for each of the eight dimensions. In 2006 this author demonstrated [Koepl2006] that it is possible to write the eight expressions of the Dirac equation in symbolic form as a direct binary split-octonion product, using only two

<sup>1</sup> Many different conventions exist around physical constants (like  $\hbar$  and  $c$ ), index notation, summation, and metric tensor  $\eta_{\mu\nu}$ . Here we just give a simplified overview, for details see the canonical literature.

factors  $\nabla\Psi = 0$ , by pairwise identifying all components. Written to split-octonion basis element  $b_{\mathbb{O}'} = \{1, i_1, i_2, i_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7\}$ , equations (3), (4), and (5) in that publication are<sup>2</sup>:

$$\Psi := (\psi_0^r, \psi_0^i, \psi_1^r, \psi_1^i, \psi_2^r, -\psi_2^i, -\psi_3^r, -\psi_3^i), \quad (2.2)$$

$$\nabla := (-m, \partial_0, 0, 0, 0, -\partial_3, \partial_2, -\partial_1), \quad (2.3)$$

$$\nabla\Psi = 0. \quad (2.4)$$

## 2.2 J. Köpflinger, V. Dzhunushaliev, M. Gogberashvili, “Emergent time from non-associative quantum theory” (2008)

In an effort to connect the initial finding into existing related context, this author was happy to collaborate for an essay contest [KDG2008]. The goal was to raise awareness of the research domain, make it more accessible and venture a thought-provoking advertisement for split-octonions in physics in general.

The essay bridged various referenced works. It also used the more modern terminology “split-octonion” and clarified where alternate terminology “hyperbolic octonions” arises from:

“Just as the octonions, the split-octonions have a multiplicative 2-form; only this time it is not Euclidean, but hyperbolic, ... The hyperbolic 2-form of split-octonions (20) ...”

Equations (27), (28), and (29) therein refer to this author’s follow-on work [Koepl2007c] that takes advantage of the simplicity of binary split-octonion product representation of the Dirac equation [Koepl2006], to introduce expansions in the form of a mixing angle  $\alpha \in \mathbb{R}$ , four-potential  $(A_0, \dots, A_3)$  and charge  $e$ . Labeling octonion basis elements explicitly as  $b_{\mathbb{O}} = (i_0, \dots, i_7)$ , these equations are:

$$\begin{aligned} \nabla_A &:= i_1 (\partial_0 - i_0 e A_0), \\ \nabla_B &:= i_5 (\partial_3 - i_0 e A_3) + i_6 (-\partial_2 + i_0 e A_2) + i_7 (\partial_1 - i_0 e A_1), \\ \nabla &:= \nabla_A + e^{i_0 \alpha} \nabla_B, \end{aligned} \quad (2.5)$$

$$\begin{aligned} \Psi_A &:= \psi_0 + i_1 \psi_1 + i_2 \psi_2 + i_3 \psi_3, \\ \Psi_B &:= -i_4 \psi_4 + i_5 \psi_5 + i_6 \psi_6 + i_7 \psi_7, \\ \Psi &:= \Psi_A + e^{i_0 \alpha} \Psi_B, \end{aligned} \quad (2.6)$$

$$(\nabla - m)\Psi = 0. \quad (2.7)$$

The reader easily verifies that this reduces to this author’s representation of the Dirac equation (2.4) when setting  $\alpha = \frac{\pi}{2}$ , the potentials  $(A_0, \dots, A_3)$  to zero, trivially relabeling the real wave function components, and pairwise identifying split-octonion basis elements  $b_{\mathbb{O}'} = \{1, i_1, i_2, i_3, -i_0 i_4, \dots, -i_0 i_7\} \equiv \{1, i_1, i_2, i_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7\}$ . The essay properly references this in “[7]”.

<sup>2</sup> Here we use the corrected 2007 form of [Koepl2006], following the corrigendum in [Koepl2007c] footnote 2. For a self-contained corrigendum with acknowledgment see e.g. ResearchGate brief DOI: 10.13140/RG.2.2.22900.41602 or this author’s personal publications web page [KoeplWWW]. The original subscript “hyp8” is omitted as it does not add value here.

### 2.3 M. Gogberashvili, A. Gurchumelia, “Split octonionic Dirac equation” (2024)

Split-octonions in [GG2024] are written to basis elements  $b_{\mathbb{O}'}$  =  $\{1, j_1, j_2, j_3, I, J_1, J_2, J_3\}$ . A wave function is initially defined in equation (4) as

$$\psi = \begin{pmatrix} \psi_4 + i\psi_7 \\ -\psi_6 + i\psi_5 \\ \psi_3 + i\psi_0 \\ \psi_1 + i\psi_2 \end{pmatrix} \quad (2.8)$$

and later (14) embedded into  $\mathbb{O}'$  as

$$\psi = \psi_0 + I\psi_4 + \sum_{n=1}^3 (j_n\psi_n + J_n\psi_{4+n}). \quad (2.9)$$

This associates one real component of the wave function with one split-octonion basis, and the function is then assumed “constant in variables  $x_0, x_5, x_6$  and  $x_7$ ”, i.e.,  $\psi : \mathbb{R}^4 \rightarrow \mathbb{R}^8$  just as in 2.2 above.

A differential operator  $\mathcal{D}$  (15) acts on the variables  $x_1, x_2, x_3$  and  $x_4 \equiv t$ ,

$$\mathcal{D} = I\partial_t - \sum_{n=1}^3 j_n\partial_n, \quad (2.10)$$

such that the split-octonionic Dirac equation can be represented as (16)

$$(\mathcal{D} - J_3m)\psi = 0, \quad (2.11)$$

which the authors then show to be “component-wise equivalent to the standard Dirac system” with the help of a computer program. The structure of (2.11) is that of a direct binary product of split-octonions, and is therefore of identical nature as (2.4), the only difference being relabeling of coefficients and vector space basis element embeddings.

For the sake of clarity, we demonstrate this explicitly. We form a new operator  $\mathcal{D}'$  by left-multiplying  $\mathcal{D}$  (2.10) with  $J_3$ , using the multiplication rules stated in [GG2024]:

$$\mathcal{D}' := J_3\mathcal{D} = j_3\partial_t + J_2\partial_1 - J_1\partial_2 - I\partial_3. \quad (2.12)$$

Right-multiplying  $\psi$  (2.8) with  $J_3$  yields:

$$\psi' := \psi J_3 \quad (2.13)$$

$$= J_3\psi_0 - j_3\psi_4 + J_2\psi_1 - j_2\psi_5 - J_1\psi_2 + j_1\psi_6 - I\psi_3 + \psi_7. \quad (2.14)$$

Because split-octonions satisfy the Moufang property  $(zx)(yz) = (z(xy))z$ , the expression

$$(\mathcal{D}' - m)\psi' = (J_3\mathcal{D} - J_3(J_3m))(\psi J_3) = (J_3((\mathcal{D} - J_3m)\psi))J_3 \quad (2.15)$$

is a structure-preserving rotation<sup>3</sup> of  $(\mathcal{D} - J_3m)\psi$  using  $J_3$ , here specifically a nonassociative generalization of group conjugation,  $aba^{-1}$ .

<sup>3</sup> For details on general rotations in the space of split-octonions see e.g. [Gogb2008].

Writing  $\mathcal{D}'$  and  $\psi'$  as split-octonions, with basis elements embedded into  $\mathbb{R}^8$  reordered to  $b'_{\mathcal{O}} = \{1, j_3, j_2, j_1, -J_3, I, -J_1, -J_2\}$ ,

$$\psi' = (\psi_7, -\psi_4, -\psi_5, \psi_6, -\psi_0, -\psi_3, \psi_2, -\psi_1), \quad (2.16)$$

$$\mathcal{D}' - m = (-m, \partial_t, 0, 0, 0, -\partial_3, \partial_2, -\partial_1). \quad (2.17)$$

this becomes identical to (2.2) and (2.3) after trivial relabeling of the real components of the wave function  $\psi'$ . Pairwise identifying  $b'_{\mathcal{O}} = \{1, j_3, j_2, j_1, -J_3, I, -J_1, -J_2\} \equiv \{1, j'_1, j'_2, j'_3, I', J'_1, J'_2, J'_3\} \equiv \{1, i_1, i_2, i_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7\}$  allows to confirm that  $b'_{\mathcal{O}}$  is indeed a split-octonion basis.

The following claims in [GG2024] are therefore untenable:

- “The novel form of the split octonionic Dirac equation . . .” has in fact previously been described by this author [Koepl2006], namely, as a *2-factor representation* that is a direct split-octonion product.
- “In conclusion, this paper has successfully derived the standard Dirac equation within the framework of split octonions” is instead an independent verification of this author’s description.
- “Methods of calculations and obtained split octonionic Dirac equation . . . are slightly different from earlier findings in [8] [*ann.*: [Gogb2006b]]. The main difference is the appearance of split octonionic imaginary unit  $J_3$  in these expressions” is both incorrect and misleading. It is incorrect because  $J_3$  generates a structure-preserving rotation, shown in equation (2.15) above. This rotation amounts to an algebra automorphism, and therefore cannot be of substantial difference in general<sup>4</sup>. It is also misleading since the vague expression “slightly different” references work [Gogb2006b] that uses a structurally different *3-factor representation* of the Dirac equation. That representation is made from a sum of factors using three split-octonion basis elements each, and is in fact leaning on prior work by different authors [DLAK1996a, DLAK1996b].

In conclusion, the authors of [GG2024] accomplished an independent verification of the finding in [Koepl2006] and demonstrated their methods in the form of a computer program. They also tied in existing other work by the authors around algebraic modeling of a Lagrangian.

### 3 Four approaches to (split-)octonions and the Dirac equation

In a broader context, there are four conceptually different approaches to the Dirac equation using split-octonions: The *2-factor*, *3-factor*, and *projection* representations using octonion basis elements natively for modeling some spacetime basis; and separately, *conventional* Dirac algebra using octonionic *spinors*<sup>5</sup>.

The *2-factor representation* expresses the Dirac equation as a direct split-octonion product [Koepl2006], as detailed above. We point out that this work, in turn, followed up on an obscure mention in [Muses1980]:

<sup>4</sup> The authors themselves investigate this e.g. in [GG2019].

<sup>5</sup> Given almost 100 years of knowledge of both the Dirac equation and split-octonions, it is well possible that further relevant work is not properly acknowledged here. Despite our best efforts, we sincerely apologize for any oversight and ask to be notified and given the chance to make amends.

“... Dirac’s equation ... although a simpler version of the equation using only 16-dimensional  $M$ -algebra [*ann.: complex octonions*] is possible ...”

No proof was given. An octonionic 2-factor representation in the complex octonions was indeed shown in [CSS2000] with the help of an  $8 \times 8$  matrix construction. That construction also includes other physical observables in the form of additional mass-like terms.

The *3-factor representation* after S. De Leo and K. Abdel-Khalek [DLAK1996a, DLAK1996b] expresses the Dirac equation as a sum of terms that generally consist of three split-octonion basis elements. It was picked up by M. Gogberashvili [Gogb2006a, Gogb2006b] by reordering and regrouping, and later followed up with an elegant mathematical treatment by R. Beradze and T. Shengelia [BS2016].

A different program has its roots in work by A. Sudbery (with K. W. Chung) [Sudb1987, CS1987], who use octonion basis elements in a  $2 \times 2$  matrix setting in a treatment primarily in terms of Lie algebra. Starting with a 10-dimensional configuration space [SM1994], T. Dray and C. A. Manogue [MD1999, DM2000] use an algebraic *projection* to retrieve faithful representations of the (associative) Dirac equation (2.1), in a way that resembles three particle families together with a sterile neutrino.

The *2-factor*, *3-factor*, and *projection* representations express the Dirac equation to split-octonion basis elements natively, which introduces nonassociativity in a way that requires clarification when connecting to canonical formulation of physical law. In contrast, octonions can also be used to model *conventional* Dirac algebra acting on certain *spinors* in an elegant algebraic way. This appeared as early as the papers of I. Bengtsson and M. Cederwall [BC1988], R. Foot and G. C. Joshi [FJ1988], and S. Marques-Bonham [MarBon1991]. Works by G. M. Dixon [Dixon1994], J. Schray and C. Manogue [SM1994], J. C. Baez and J. Huerta [BH2010], and N. Furey [Furey2012, Furey2016, Furey2025] are just a few prominent modern examples in this investigation towards driving physical laws from the nature of numbers.

This review section hopes to help clear up recurring confusion in the literature about the proper context of one’s work when it comes to octonions and the Dirac equation. To give an example, the projection approach from Dray and Manogue [MD1999, DM2000] appears to have been picked up recently in an interesting work by J. C. Vélez Quiñones [VQ2022]; however, that author quotes De Leo and Abdel-Khalek [DLAK1996b] instead, which does not appear to be used in that paper. Ironically, in a recent work coauthored by Manogue and Dray [MDW2022] the authors cite their projection approach [MD1999, DM2000], even though no such projection is used or needed; instead, their construction of Lie algebras from one-sided octonionic multiplication would lend itself towards this author’s direct product [Koepl2006] when looking for spacetime representations in not necessarily associative subspaces. Confusion aside, references to the Dirac equation in these examples are made “in passing” and not substantial for the respective findings.

Finally, we would like to point to recent mathematical methods for native split-octonionic analysis [LZ2024, HRS2025, KLL2025, KLL2025weyl, LR2025, Lopat2026], which could make many of the above symbolic expressions on nonassociative algebra analytically tractable. This includes a large body of contemporary work [CTÖD2008, Weng2009, TKD2012, Demir2013, DTK2013, TKD2014, CSN2015, MBCM2020, Koepl2023a, Koepl2023b, Lasen2023, KHG2025], to mention just a few, with apologies for other original approaches that were inadvertently missed.

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