Dirac equation on hyperbolic octonions — a guided tour of the split-octonion Dirac equation

A research digest of Köplinger (2006)

Abstract

This is a guided tour of the paper **J. Köplinger, "Dirac equation on hyperbolic octonions",** ***Appl. Math. Comput.*** **182 (2006) 443–446** (DOI: 10.1016/j.amc.2006.04.005), with corrections from the 2007 corrigendum (ResearchGate DOI: 10.13140/RG.2.2.22900.41602). The paper rewrites the 4×4-matrix Dirac equation as a single non-associative product $\nabla\Psi = 0$ on the eight-dimensional *split-octonions* (the paper's "hyperbolic octonions"), and embeds that identity in a sixteen-dimensional *complex-octonion* ("conic-sedenion") ambient where a continuous hypernumber rotation interpolates between a split-octonion and an octonion form of the equation. The digest walks through the construction, the corrigendum-corrected expressions, the proof-by-bookkeeping, the hypernumber-rotation symmetry, and the paper's relationship to Charles Musès' 1980 conjecture. A review section surveys the four conceptually distinct approaches to (split-)octonions and the Dirac equation in the wider literature, and a dedicated section lays out the priority and attribution record against the 2024 Gogberashvili–Gurchumelia paper. Follow-on work is referenced in the body. The digest takes the published journal paper, together with its 2007 corrigendum, as the authoritative reference.

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1 At a glance

The Dirac equation is the fundamental relativistic wave equation for a free spin-$\frac{1}{2}$ particle. In its standard form it is a matrix relation between a four-component complex wave function and a 4×4 differential operator. This paper shows that the whole of that relation can be written as a **single non-associative product** on an eight-dimensional hypernumber algebra — the **split-octonions**.

Concretely, if one packs the wave function into an eight-component split-octonion Ψ and the derivative-and-mass content into a second split-octonion ∇, then the Dirac equation is exactly

$$\nabla \cdot \Psi \;=\; 0. \tag{1.1}$$

The paper then embeds this identity inside a 16-dimensional ambient algebra — the **complex octonions** — where a continuous rotation in a distinguished hypernumber plane continuously deforms the split-octonion Dirac equation into an **octonion** counterpart, showing that "geometry" can be treated as an angle rather than a switch.

2 Who this digest is for

Readers who know linear algebra and basic complex analysis, are comfortable with the idea that the Dirac equation is a 4×4 matrix relation on a complex four-component object, and are willing to meet two new algebras along the way — *octonions* (non-associative, eight real dimensions) and *complex octonions* (sixteen real dimensions, non-associative, non-alternative). No prior exposure to Musean hypernumbers is assumed.

3 Note on terminology

The original paper, and the author's other early *Applied Mathematics and Computation* papers (2006–2007), use the *Musean hypernumber* vocabulary then in active exchange among the small community around Charles Musès and Kevin Carmody. In that vocabulary:

Name used in the paper	Conventional modern name	Meaning
hyperbolic octonions	**split-octonions**	eight-real-dimensional composition algebra, signature $(4, 4)$
conic sedenions	**complex octonions**	octonions with complex coefficients; sixteen real dimensions
circular octonions	**octonions**	the ordinary (Cayley) octonions

The two vocabularies are synonymous; they name the same algebras. The original terminology was adopted in the context of the research community that fostered this work at the time, and the author is grateful for the help, encouragement, and correspondence received in those early days. The task of this digest is to place the work in the wider modern literature, so from here on the conventional names are used throughout — *split-octonion*, *complex octonion*, *octonion* — without further comment. Readers who wish to match section by section against the original paper should simply apply the dictionary above.

4 The Dirac equation in one page

Paul Dirac's 1928 equation is a first-order relativistic wave equation for a free spin-$\frac{1}{2}$ particle — an electron, for example. The wave function is a four-component complex object,

$$\Psi = (\psi_0, \psi_1, \psi_2, \psi_3), \qquad \psi_\mu = \psi_\mu^r + i\,\psi_\mu^i \in \mathbb{C}, \tag{4.1}$$

and the ψ_μ depend on space (x_1, x_2, x_3) and time x_0. The equation can be written as a 4×4 matrix equation acting on Ψ; one common form (the "Dirac representation", with $\hbar = c = 1$) is

$$\begin{pmatrix} -m + i\partial_0 & 0 & -i\partial_3 & -i\partial_1 - \partial_2 \\ 0 & -m + i\partial_0 & -i\partial_1 + \partial_2 & i\partial_3 \\ i\partial_3 & i\partial_1 + \partial_2 & -m - i\partial_0 & 0 \\ i\partial_1 - \partial_2 & -i\partial_3 & 0 & -m - i\partial_0 \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4.2}$$

Here m is the particle's rest mass and $\partial_\mu := \partial/\partial x_\mu$. This matrix form is the standard "one $\mathbb{C}$, many matrices" way of writing Dirac, and it works beautifully. The question this paper asks is different: **can the same relation be written as a single algebraic product, in one step, on a richer number system?**

5 Beyond complex numbers: a brief hypernumber tour

The real numbers $\mathbb{R}$ extend to the complex numbers $\mathbb{C}$ by adjoining an element i with $i^2 = -1$. Repeat the process:

- Add two more anti-commuting square roots of -1 → **quaternions** $\mathbb{H}$, four real dimensions, non-commutative but associative.
- Add three more → **octonions** $\mathbb{O}$, eight real dimensions, non-commutative *and* non-associative, but *alternative* (products of two elements from two directions still associate).
- Repeat once more → **sedenions**, sixteen real dimensions, losing even alternativity and acquiring zero divisors.

Each step preserves the key structural move: form a new algebra whose basis is the basis of the old algebra together with a "fresh" imaginary unit and the products of that unit with everything already there (the *Cayley–Dickson doubling*). All the algebras so constructed use only square roots of -1.

Split sisters. A parallel family replaces some of the square roots of -1 with square roots of $+\mathbf{1}$ (so-called *hyperbolic* units, ε with $\varepsilon^2 = +1$). The two-dimensional version of this is the algebra of **split-complex numbers**; the eight-dimensional version — with four hyperbolic units alongside the ordinary four imaginary units — is the **split-octonions**. Octonions and split-octonions both sit as eight-dimensional subalgebras inside a common sixteen-dimensional parent, the **complex octonions** (octonions over $\mathbb{C}$; equivalently, the Musean *conic sedenions* [Muses1980, Carmody1988, Carmody1997], as carefully distinguished from the Cayley–Dickson sedenions of [ImaedaImaeda2000]). The complex-octonion structure introduces one extra imaginary unit — the complex i itself — which anti-commutes with the three octonion imaginary units it is multiplied against and commutes with the rest; this is what makes the complex octonions neither alternative nor associative while still being a composition algebra.

6 The split-octonions in one screen

A **split-octonion** is an eight-real-component number of the form

$$z = c_1 \cdot 1 + c_{i_1}\, i_1 + c_{i_2}\, i_2 + c_{i_3}\, i_3 + c_{\varepsilon_4}\, \varepsilon_4 + c_{\varepsilon_5}\, \varepsilon_5 + c_{\varepsilon_6}\, \varepsilon_6 + c_{\varepsilon_7}\, \varepsilon_7, \tag{6.1}$$

abbreviated

$$z = (c_1,\, c_{i_1},\, c_{i_2},\, c_{i_3},\, c_{\varepsilon_4},\, c_{\varepsilon_5},\, c_{\varepsilon_6},\, c_{\varepsilon_7}). \tag{6.2}$$

The three elements i_1, i_2, i_3 are imaginary (square to -1); the four elements $\varepsilon_4, \dots, \varepsilon_7$ are hyperbolic (square to $+1$); 1 is the unit. The basis anticommutes pairwise off the real axis. The multiplication table is fixed by the split-octonion composition identity

$$\|z\| = \left(c_1^2 + c_{i_1}^2 + c_{i_2}^2 + c_{i_3}^2\right) - \left(c_{\varepsilon_4}^2 + c_{\varepsilon_5}^2 + c_{\varepsilon_6}^2 + c_{\varepsilon_7}^2\right), \tag{6.3}$$

$$\|z \cdot w\| = \|z\| \cdot \|w\|. \tag{6.4}$$

Products are well-defined but non-associative. Many equivalent choices of basis and multiplication table exist; the split-octonion algebra itself is unique up to isomorphism [Jacob1958, Okubo1995, Baez2002, ConwaySmith2003].

The ordinary complex imaginary i is identified with the basis element i_1 once we move into the split-octonion setting. This identification is fixed throughout the paper.

7 Packing the Dirac objects into split-octonions

The paper's central construction is to collapse the four complex components of the Dirac wave function into the eight real coefficients of a split-octonion. The chosen packing is

$$\Psi \;:=\; \big(\psi_0^r,\, \psi_0^i,\, \psi_1^r,\, \psi_1^i,\, \psi_2^r,\, -\psi_2^i,\, -\psi_3^r,\, -\psi_3^i\big). \tag{7.1}$$

The rule behind the signs: for each μ, the real part ψ_μ^r sits in slot $c[b_\mu]$ and the imaginary part ψ_μ^i sits in slot $c[b_\mu \cdot i_1]$, where b_μ runs through the eight basis elements in the order given. The final three slots carry minus signs because of how the products $b_\mu \cdot i_1$ land in the split-octonion multiplication table.

The derivative-and-mass content of the Dirac matrix is packed into a second split-octonion

$$\nabla \;:=\; \big(-m,\, \partial_0,\, 0,\, 0,\, 0,\, -\partial_3,\, \partial_2,\, -\partial_1\big). \tag{7.2}$$

Note on the 2007 corrigendum. The published paper used a multiplication-table convention that mapped the classical octonion element "ℓ" to $-i_4$ instead of $+i_4$, as pointed out by Süleyman Demir of Anadolu University. The corrigendum restores the convention used in the Carmody references, which flips the sign of the ψ_2^r slot of Ψ and of the ∂_0 slot of ∇ relative to the published definitions. **The two forms (7.1) and (7.2) are the corrigendum-corrected ones** — this digest uses the corrected convention throughout. All downstream results of the paper remain unchanged in interpretation. The corrigendum is available as ResearchGate DOI: 10.13140/RG.2.2.22900.41602 [KoeplCorr2007].

8 The main result: Dirac as a single split-octonion product

With Ψ and ∇ in hand, the central claim of the paper is a single line:

$$\nabla \cdot \Psi \;=\; 0 \quad\Longleftrightarrow\quad \text{the Dirac equation.} \tag{8.1}$$

That is, the product on the left-hand side — computed using the eight-dimensional split-octonion multiplication — is the zero split-octonion **if and only if** the original Dirac equation is satisfied. What was a 4×4 matrix relation on four complex numbers is an eight-component product on a single split-octonion.

9 How the proof works

Because the claim is an "if and only if" between an 8-component vector equation and a 4-component complex equation, the proof is essentially bookkeeping. In outline:

1. **Expand the Dirac equation into eight real equations.** Each of the four complex Dirac equations (rows of the matrix relation in §4) splits into a real part and an imaginary part. That produces eight real equations, each of which sets a linear combination of $m \cdot \psi$, $\partial_\mu \cdot \psi$, and so on to zero.

2. **Compute $\nabla \cdot \Psi$ component by component.** The split-octonion product returns an eight-component split-octonion. Working through the products of each basis pair ($i_1 \cdot \varepsilon_6$, $\varepsilon_4 \cdot i_3$, …) using the fixed multiplication table, each of the eight output components is a specific linear combination of $m \cdot \psi$ and $\partial_\mu \cdot \psi$ terms.
3. **Match components.** Each of the eight output components of $\nabla \cdot \Psi$ matches, term for term, the left-hand side of one of the eight real Dirac equations from step 1 — either exactly, or with an overall sign flip. Setting the split-octonion product to zero therefore reproduces all eight equations simultaneously.

The verification is an exercise in following signs through the multiplication table. Nothing deeper is needed; the striking thing is only that the coefficient choices in Ψ and ∇ line up so cleanly that no linear recombination or basis change is required at the end.

10 Hypernumber rotation as symmetry

Having established the Dirac equation as a split-octonion product, the paper's second act embeds it in the larger **complex octonion** algebra. A complex octonion is a sixteen-real-component hypernumber with the octonion basis

$$\{1, i_1, i_2, \ldots, i_7\} \tag{10.1}$$

together with the complex unit i_0 (which the original paper calls the conic imaginary) and the cross-products $i_0 \cdot i_k$. Inside it, both the octonions (built on imaginary units alone) and the split-octonions (where the last four units are hyperbolic, $\varepsilon_k = i_0 \cdot i_{3+k}$) sit as eight-dimensional subalgebras. The complex unit i_0 generates rotations mixing the two subalgebras.

The construction is a three-step recipe:

1. Write the split-octonion operator and wave function as a sum of two pieces: one piece that "looks the same" in both geometries and one piece that carries the sign/geometry distinction. Call these pieces ∇_{Q1}, ∇_{Q2} for the operator and Ψ_{Q1}, Ψ_{Q2} for the wave function.
2. In the complex-octonion ambient algebra, glue them back together with a complex-like phase factor $\exp(\alpha \cdot i_0)$:

$$\nabla_{\text{con}} := \nabla_{Q1} + \exp(\alpha \cdot i_0) \cdot \nabla_{Q2}, \tag{10.2}$$

$$\Psi_{\text{con}} := \Psi_{Q1} + \exp(\alpha \cdot i_0) \cdot \Psi_{Q2}. \tag{10.3}$$

3. Evaluate at two special angles:
   - At $\alpha = \pi/2$, the exponential becomes i_0, and the construction reproduces the split-octonion Dirac equation of §7.
   - At $\alpha = 0$, the exponential becomes 1, and the construction reproduces an analogous equation on the **octonions**.

In between, the complex-octonion relation $\nabla_{\text{con}} \cdot \Psi_{\text{con}} = 0$ interpolates continuously between the two geometries. In physicists' language: the angle α acts as a **mixing parameter** between the familiar Minkowski-style Dirac relation and a hypothetical "octonionic" (Euclidean) counterpart, and the rotation generated by $\exp(\alpha \cdot i_0)$ is a new candidate class of symmetry transformation — a *genuine hypernumber rotation* — not expressible by any ordinary complex-valued matrix acting on Ψ.

The paper is careful to stress that this second part is method, not physics: the author is demonstrating that hypernumber arithmetic *can* be used to broaden the symmetry operations available to the Dirac equation, without claiming that the specific interpolation above is the one nature chooses. The subsequent papers in the series [Koepl2007a, Koepl2007b, Koepl2007c] take up the octonionic-geometry side and explore gravity-like and electromagnetism-like counterparts to the Dirac relation, and a corresponding relativity principle.

11 Relationship to Charles Musès

Musès' conjecture. The paper explicitly positions itself as a **proof** of a statement by the American philosopher and independent mathematician **Charles Musès** (1919–2000). In [Muses1980], Musès asserted that "*... a simpler version of the equation using only 16-dimensional M-algebra is possible...*" — his name for what is now called complex octonions. Musès did not give the construction; he conjectured that it existed.

The author's stance. This paper adopts the Musean hypernumber framework **as its working foundation** — complex octonions and their subalgebras are taken as given — and then delivers the specific construction Musès asserted was possible. The mode here is adopt-and-extend rather than critique-and-reconstruct: Musès is cited respectfully as the source of the conjecture and the originator of the relevant 16-dimensional algebra, and the paper proves the conjecture within that framework.

The more detailed mathematical machinery — notation, multiplication tables, and the careful distinction between complex octonions and other 16-dimensional systems (such as the Cayley–Dickson sedenions of Imaeda & Imaeda [ImaedaImaeda2000]) — comes from the follow-up work of **Kevin Carmody** [Carmody1988, Carmody1997], whose conventions the paper uses throughout. The acknowledgment in the paper is to Carmody "for his gracious help with hypernumber arithmetics".

12 Four approaches to (split-)octonions and the Dirac equation

The construction of §7 is not the only way to bring octonion-family algebras into the Dirac equation. It is one of four conceptually distinct approaches that have been developed in the literature. The first three use octonion basis elements natively to model spacetime; the fourth uses octonions to carry conventional Dirac algebra acting on octonionic spinors.

(i) The 2-factor (direct-product) representation. The present paper. The Dirac equation is written as a single direct product $\nabla \cdot \Psi = 0$ on split-octonions. The 16-dimensional complex-octonion version — with additional mass-like terms — had earlier been obtained by F. Colombo, I. Sabadini, and D. C. Struppa [CSS2000] via an 8×8 matrix construction. The split-octonion 2-factor form given here is the one directly applicable to the hyperbolic 4+4 signature needed for the physical Dirac equation.

(ii) The 3-factor representation. S. De Leo and K. Abdel-Khalek [DLAK1996a, DLAK1996b] write the Dirac equation as a sum of terms, each a product of **three** octonionic basis elements. This structure was later reorganized by M. Gogberashvili [Gogb2006a, Gogb2006b] and followed up with an elegant mathematical treatment by R. Beradze and T. Shengelia [BS2016].

(iii) The projection representation. A. Sudbery [Sudb1987] and K. W. Chung & A. Sudbery [CS1987] use octonion basis elements in a 2×2 matrix setting, with the emphasis on the Lie-algebra structure. Building on this, J. Schray and C. A. Manogue [SM1994] start from a 10-dimensional configuration space, and C. A. Manogue and T. Dray [MD1999, DM2000] use an algebraic projection to recover faithful representations of the associative Dirac equation. The resulting structure resembles three particle families together with a sterile neutrino.

(iv) Octonionic spinors in conventional Dirac algebra. The 2-factor, 3-factor, and projection approaches all put octonion basis elements natively into the spacetime description, which introduces non-associativity that needs clarification when matching to canonical physical law. The fourth approach keeps the Dirac algebra conventional (associative matrices acting on spinors) and lets the **spinor** take values in an octonion-related algebra. This goes back at least to I. Bengtsson and M. Cederwall [BC1988], R. Foot and G. C. Joshi [FJ1988], and S. Marques-Bonham [MarBon1991]. Influential modern developments include G. M. Dixon [Dixon1994], J. C. Baez and J. Huerta [BH2010], and C./N. Furey [Furey2012, Furey2016, Furey2025].

Native split-octonionic analysis. The native approaches (i)–(iii) would benefit from direct tools of split-octonionic analysis, which has become an active subject in its own right over the past few years. Relevant threads include A. Lopatin and A. N. Zubkov [LZ2024]; A. Lopatin and A. N. Ryballov [LR2025]; A. Lopatin [Lopat2026]; Q. Huo, G. Ren, and I. Sabadini [HRS2025]; and the discrete-octonionic- analysis program of R. S. Kraußhar, A. Legatiuk, and D. Legatiuk [KLL2025, KLL2025weyl]. These methods promise to render many symbolic non-associative expressions analytically tractable.

Contemporary applications. A substantial body of work applies split- and hyperbolic-octonion formulations to Maxwell, Proca, gravity, and related field equations: N. Candemir, M. Tanışlı, K. Özdaş, and S. Demir [CTOD2008]; Z. Weng [Weng2009]; M. Tanışlı, M. E. Kansu, and S. Demir [TKD2012, TKD2014]; S. Demir [Demir2013]; S. Demir, M. Tanışlı, and M. E. Kansu [DTK2013]; B. C. Chanyal, V. K. Sharma, and O. P. S. Negi [CSN2015]; S. Marques-Bonham, B. C. Chanyal, and R. Matzner [MBCM2020]; A. Lasenby [Lasen2023]; and the author's own follow-on including J. Köplinger [Koepl2023] and J. Köplinger, M. Habeck, and P. Goyal [KHG2025].

13 Extensions and follow-on by the author

The paper is the first of a four-paper sequence by the author that applies split-octonion and complex-octonion algebra to the Dirac equation, gravity, electromagnetism, and relativity. The three subsequent papers in that sequence are [Koepl2007a, Koepl2007b, Koepl2007c].

Beyond the AMC sequence, further direct developments include:

- **J. Köplinger, V. Dzhunushaliev, M. Gogberashvili** [KDG2008], "Emergent time from non-associative quantum theory" (FQXi essay, 2008) — uses the same split-octonion framework and invites the reader to speculate and dream about time emergence from non-associative quantum theory.

- **J. Köplinger** [Koepl2023], "Phenomenology from Dirac equation with Euclidean–Minkowskian 'gravity phase'," a 2023 continuation of the same research arc that models configuration space on 4×4 matrices over the bicomplex numbers $\mathbb{C} \oplus \mathbb{C}$, carrying the "mixing-angle" idea of §10 above into an explicit phenomenological framework.

- **J. Köplinger, M. Habeck, P. Goyal** [KHG2025], "Operational reconstruction of Feynman rules for quantum amplitudes via composition algebras" (2025) — a reconstruction of quantum amplitudes from operational principles; the present paper's split-octonion Dirac construction may become relevant here once spacetime — and with it dynamics and interaction — is added to the operational model. Which of the approaches surveyed in §12 fits best is an open question, which is part of why a careful accounting of the alternatives seems worth keeping on hand.

14 The 2024 Gogberashvili–Gurchumelia paper: priority and attribution

A 2024 paper by M. Gogberashvili and A. Gurchumelia [GG2024] presents a split-octonion Dirac equation of exactly the 2-factor direct-product type given in §7 of the present paper, but claims "the novel form of the split octonionic Dirac equation" without citing either the 2006 construction or the intervening 2008 collaboration that transmitted it. This section lays out the record. The author has set out the full argument in a separate 2026 research brief [KoeplRefut2026]; the essentials are summarized here.

The timeline.

- **2006.** The present paper [Koepl2006] gives the 2-factor split-octonion Dirac equation $\nabla \cdot \Psi = 0$ with the explicit forms of §7–§8 above.

- **2008.** J. Köplinger, V. Dzhunushaliev, and M. Gogberashvili co-author "Emergent time from non-associative quantum theory" [KDG2008], which explicitly uses and cites the 2-factor split-octonion Dirac equation from the 2006 paper (equations (27), (28), (29) of that essay reduce to $\nabla \cdot \Psi = 0$ at $\alpha = \pi/2$ after pairwise basis identification). The essay also introduces the modern term "split-octonion" and explains its relation to "hyperbolic octonion". M. Gogberashvili is a co-author of this essay and is therefore familiar with the 2006 result.

- **2024.** The GG2024 paper [GG2024] presents a split-octonion Dirac equation $(D - J_3 m)\psi = 0$ with basis $\{1, j_1, j_2, j_3, I, J_1, J_2, J_3\}$ and $D = I\partial_t - \sum j_n \partial_n$, and claims novelty. Neither the 2006 paper nor the 2008 essay is cited.

Why the "novel form" is equivalent to §7. The GG2024 equation differs from the 2006 equation only by a structure-preserving rotation of the split-octonion basis. Left-multiplying D by J_3 and right-multiplying ψ by J_3 produces an operator $D' = J_3 D$ and wave function $\psi' = \psi J_3$ that, with the reordered basis $\{1, j_3, j_2, j_1, -J_3, I, -J_1, -J_2\}$, match the corrigendum-corrected 2006 expressions

$$\psi' = (\psi_7,\, \psi_4,\, -\psi_5,\, \psi_6,\, -\psi_0,\, -\psi_3,\, \psi_2,\, -\psi_1), \tag{14.1}$$

$$D' - m = (-m,\, \partial_t,\, 0,\, 0,\, 0,\, -\partial_3,\, \partial_2,\, -\partial_1). \tag{14.2}$$

The key identity is the **Moufang property** of the (split-)octonions,

$$(z\,x)(y\,z) \;=\; \big(z(x\,y)\big)z, \tag{14.3}$$

which makes simultaneous left- and right-multiplication by a fixed unit z — here $z = J_3$ — an algebra automorphism. The GG2024 form is thus the image of the 2006 form under a single split-octonion automorphism. The claim that the appearance of J_3 in the new form constitutes a "main difference" from earlier octonionic Dirac work therefore does not survive examination: an automorphism of the algebra cannot be a substantive difference.

What GG2024 actually contributes. Read without the novelty claim, the GG2024 paper is an independent **verification** of the 2006 result, carried out by a different route and supported by a computer program that exhibits the component-wise equivalence with the standard Dirac system. The paper also ties the construction to the authors' Lagrangian-side work. These contributions stand on their own and are welcome. The problem is the framing — that the split-octonion Dirac equation is presented as a new discovery when the 2-factor form has been in the literature since 2006 and was co-authored with one of the 2024 authors in 2008. The refutation brief [KoeplRefut2026] argues, and this digest agrees, that the 2024 paper is best read as **independent verification, not independent discovery**.

How to cite the situation. Readers working on split-octonion Dirac equations who are pointed to [GG2024] should be aware that the equation type given there is the 2-factor form of the present paper. When citing, it is appropriate to cite the 2006 paper [Koepl2006] as the origin of the 2-factor split-octonion Dirac equation; the 2024 paper may then be cited as an independent verification and Lagrangian-side extension, alongside the refutation brief [KoeplRefut2026] for the full side-by-side comparison.

15 Reception in the literature

The paper and its 2007 siblings have been picked up in the subsequent literature on split-octonion and hyperbolic-octonion applications to classical and quantum field theory, including the Anadolu-University body of work on Maxwell, Proca, and related equations in split-octonion form cited in §12 [CTOD2008, TKD2012, TKD2014, Demir2013, DTK2013], as well as related developments by Chanyal, Sharma, and Negi [CSN2015], Marques-Bonham, Chanyal, and Matzner [MBCM2020], and others.

The paper's construction — a single non-associative product replacing a matrix equation — remains a clean, self-contained result that other workers can pick up without extra machinery, and the native split-octonionic analysis tools of Lopatin & Zubkov [LZ2024], Lopatin & Ryballov [LR2025], Huo, Ren, & Sabadini [HRS2025], and Kraußhar & the Legatiuks [KLL2025, KLL2025weyl] (see §12) are beginning to make that pickup analytically richer.

16 How to read the original

The paper is compact (four journal pages) and mostly calculation. It was published in *Applied Mathematics and Computation* **182** (2006) 443–446, DOI: 10.1016/j.amc.2006.04.005 [Koepl2006]. For preprints and personal versions, see the author's web site [KoeplWWW]; the author's ResearchGate profile [ResearchGateJK] mirrors the preprints. The 2007 corrigendum — a one-page author's note reconciling the multiplication-table convention with Carmody's references and giving the corrected sign assignments — is available as ResearchGate DOI: 10.13140/RG.2.2.22900.41602 [KoeplCorr2007]; the reader should consult it before using the explicit expressions from the published paper.

The paper is organized as follows:

Section	Content
§1 Introduction	Musès' claim; the choice of split-octonions (the paper's "hyperbolic octonions")
§2 Dirac on split-octonions	Packing of Ψ and ∇; the product identity; eight-equation proof
§3 Hypernumber rotation	Embedding in complex octonions (the paper's "conic sedenions"); the $(1, i_0)$ rotation; octonionic counterpart
References	Musès 1980; Carmody 1988, 1997; Imaeda & Imaeda 2000

17 About the author

Jens Köplinger is an independent researcher in mathematical physics. His personal page with publications, preprints, fractal galleries, and audio renderings is [KoeplWWW]; his ResearchGate profile is [ResearchGateJK]. ORCID: 0009-0003-6011-5598.

18 How to cite

For the original research, cite the paper together with its corrigendum:

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>
> J. Köplinger, "Corrigendum to *Dirac equation on hyperbolic octonions*", 1 November 2007. ResearchGate DOI: 10.13140/RG.2.2.22900.41602.

For this digest, please refer to the ResearchGate entry on the author's profile [ResearchGateJK].

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