Beyond the Complexes:
Toward a lattice based number system.

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"As time goes on, it becomes increasingly evident that the rules which the mathematicians find interesting are the same as those which Nature has chosen."
Paul A. M. Dirac

In this talk

Intro: Research statement

Part 1: Lattice-numbers in one dimension

Part 2: Lattice-numbers in 2, 4, and 8 dimensions

Outlook

Why any of this?

Fact (*): Although incompatible in principle, Quantum Mechanics and General Relativity model different aspects of the same reality.

(*) It is customary for scientific facts to change over time.

Speculation: A compatible model exists but hasn't been found yet.

Why any of this?

Fact: It is essentially impossible to model most quantum systems without Complex numbers.

Speculation: Quantum Mechanics and Complex numbers are like fish and water. Simple but powerful systems in mathematics and models of nature somehow relate both ways.

Very broad research statement: Find or develop a number system that enables or makes possible a compatible model for all aspects of Quantum Mechanics and General Relativity.

Guiding principle: Theoretical reductionism, to model these aspects similar to but simpler (*) than today's models. The number system to be found or developed must be similar to but simpler than arithmetic in use today.

(*) Here, "simpler" means conceptually simpler, not necessarily simpler to calculate.

In this talk

Specific research statement: Develop an arithmetic/algebra and topology on a set of numbers represented by digits on lattice points.

Reference: Cashier's vision *[1]*: "$1 + 1 = 2$, a step in the wrong direction?". Number and arithmetic dual to one another, reflect dualities observed in Quantum Mechanics.

Part 1

Lattice-numbers in one dimension.

Real / 2-adic number representation

$$
\begin{aligned}
a &\equiv \quad \ldots \quad a_2 \quad a_1 \quad a_0. \quad a_{-1} \quad a_{-2} \quad \ldots \\
&\equiv \sum_i a_i 2^i
\end{aligned}
$$

One-dimensional lattice-number:

$$\ldots \quad a_2 \quad a_1 \quad \underline{\overline{a_0}} \quad a_{-1} \quad a_{-2} \quad \ldots$$

($\underline{\overline{\square}}$ is "origin").

Conventional addition

Decimal: $2 + 3 = 5$

Binary: $10 + 11 = 101$

Lattice: $1 \;\; \underline{\overline{0}} + 1 \;\; \underline{\overline{1}} = 1 \;\; 0 \;\; \underline{\overline{1}}$

Lattice-number addition AX+1D

Conventional addition is binary lattice-number morphism:

Argument: $1\ \overline{\underline{0}}\ ,\ 1\ \overline{\underline{1}}$

Result: $1\ 0\ \overline{\underline{1}}$

Notation: AX+1D on L1

- "A" **a**ddition type: same coordinates
- "X" pairwise **X**OR
- "+1D" carry-over to **next** neighbor (**d**irected)

AX+1D on L1

Unique additive inverse exists, e.g.:

$$1 \quad \underline{\overline{0}} + \ldots \quad 1 \quad \ldots \quad 1 \quad 1 \quad \underline{\overline{0}} = \underline{\overline{0}}$$

(see 2-adics). In general, additive inverse is the dual number, e.g.:

$$\begin{aligned} 1 \quad \underline{\overline{0}} \quad &+ \quad \ldots \quad 1 \quad \ldots \quad 1 \quad 0 \quad \underline{\overline{1}} \quad 1 \quad \ldots \quad 1 \quad \ldots \\ &= \quad \underline{\overline{0}} \end{aligned}$$

Requires existence of infinite limits.

Conventional multiplication

Decimal: $2 * 3 = 6$

Binary: $10 * 11 = 110$

Lattice: $1 \ \underline{\overline{0}} * 1 \ \underline{\overline{1}} = 1 \ 1 \ \underline{\overline{0}}$

Lattice-number multiplication MX+1D

Conventional multiplication is binary lattice-number morphism "MX+1D on L1":

Argument: $1\ \underline{\overline{0}}\ ,\ 1\ \underline{\overline{1}}$

Result: $1\ 1\ \underline{\overline{0}}$

- "M" **m**ultplication type: pair coordinate addition
- "X" **X**OR at image coordinate
- "+1D" carry-over to **next** neighbor (**d**irected)

MX+1D is not well-defined for infinite nonrepeating sequences $(a_k)_{k\in\mathbb{N}}$:

$$\ldots \quad (a_k) \quad \ldots \quad \underline{\overline{0}} \quad * \quad \underline{\overline{0}} \quad \ldots \quad (a_k) \quad \ldots \qquad = \quad ??$$

One way out: Bounded lattice numbers with finite n where

$$k > n \Longrightarrow a_k = a_{k+1}.$$

Examples: Real or 2-adic numbers.

Lattice-number exponentiation EX+1D

New lattice-number exponentiation "EX+1D on L1" ($^\vee$) defined like MX+1D but using lattice coordinate multiplication ("E" **e**xponentiation type).

$$\text{Lattice:}\quad 1\ \ \underline{\overline{0}}\ {}^\vee\, 1\ \ \underline{\overline{1}}\ = \ 1\ \ \underline{\overline{1}}$$

$$\text{Binary: } 10\,{}^\vee\, 11 = 11$$

$$\text{Decimal: } 2\,{}^\vee\, 3 = 3$$

Lattice: $1 \quad 0 \quad \underline{\overline{0}} \stackrel{\vee}{} 1 \quad 0 \quad 0 \quad \underline{\overline{0}} =$
$1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \underline{\overline{0}}$

Binary: $100 \stackrel{\vee}{} 1000 = 1000000$

Decimal: $4 \stackrel{\vee}{} 8 = 64$

EX+1D on L1 has subspace as conventional

$$a \stackrel{\vee}{} b := 2\hat{}\,[(\log_2 a)\,(\log_2 b)]$$

But generally is different.

Summary so far

AX+1D: real and 2-adic addition

MX+1D: real *or* 2-adic multiplication

EX+1D: subspace exist with real or 2-adic
$2\hat{}\,[(\log_2 a)\,(\log_2 b)]$

Note

Invertibility of EX+1D depends on invertibility of underlying lattice coordinate (vector) multiplication.

Part 2

Lattice-numbers in 2, 4, and 8 dimensions

Rays

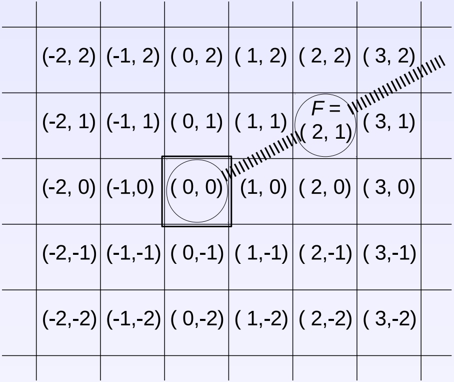
$$\mathrm{Ray}\,(F) := F\mathbb{N}_0$$

$$\{F\} := \{(a, b)|$$

$$\gcd(a, b) = 1\}$$

Example (right):

$$F = (2, 1)$$

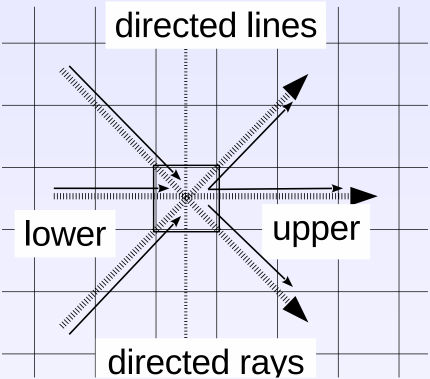


Directed rays, directed lines

Directed ray, directed line:

Direction of carry-over by convention (*)

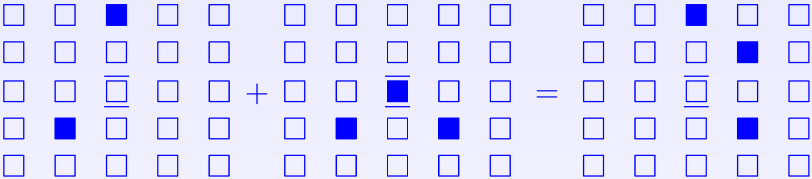
Morphisms AX+1D, MX+1D and EX+1D work on any lattice.



(*) There is an edge case for carry-overs through the origin.

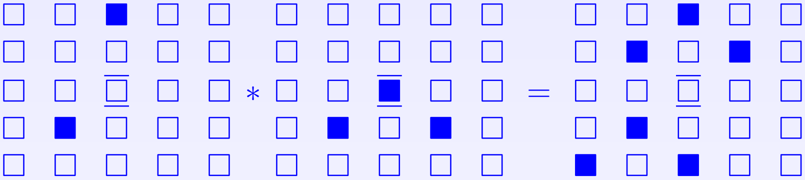
Example of AX+1D on L2

Example of AX+1D on L2:



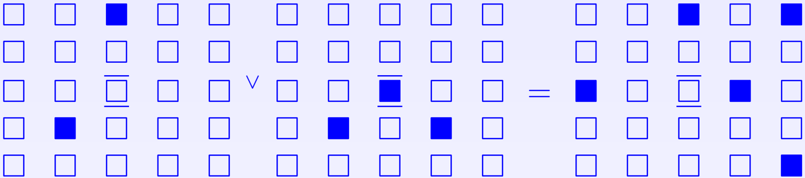
Invertible, commutative, nonassociative (and generally nonalternative) due to carry-over through the origin.

Example of MX+1D on L2:



Invertible, commutative, nonassociative / nonalternative.

Example of EX+1D on L2:



Here: Lattice (vector) coordinate multiplication is integral $\mathbb{C}$.

AX+1D, MX+1D, and EX+1D are *invertible* on lattices over integral normed division algebras. Straightforward in two ($\mathbb{C}$) and four ($\mathbb{H}$) dimensions.

In 8D ($\mathbb{O}$) Geoffrey Dixon's integral octonions *[2]*:

- Two dual E_8 lattices in $\mathbb{R}^8$: Ξ^{odd} and Ξ^{even}
- $X \in \Xi^{odd}, A, B \in \Xi^{even} \Rightarrow \left(AX^{\dagger}\right)(XB) \in \Xi^{even}$

E8 lattice-numbers are simple!

Similar to, but simpler than, conventional octonions. Morphisms Ξ^{odd} and numbers Ξ^{even} are duals. Rich configuration space.

Lattice coordinates $\{c_i\}$, lattice-numbers A, B with digits $\{a_{c_i}\}, \{b_{c_i}\}$. Then:

$$\begin{aligned} d_i(A,B) &:= (a_{c_i} \oplus b_{c_i}) \exp(-|c_i|), \\ d(A,B) &:= \sum_i d_i(A,B). \end{aligned}$$

- Metric space, Hausdorff
- ε neighborhood is essentially the entire lattice
- Many other $d(A,B)$ possible
- Normed "vector space" ??

Challenges and outlook

- Many different morphisms, algebraic properties
- Many, many formal proofs to do
- Nonrepeating sequences
- Carry-over through the origin on nonchiral lattices
- Chiral lattices, e.g., Leech lattice?
- Differential calculus
- Norm?

Thank you!

References

[1] J. Köplinger, J. A. Shuster, "$1 + 1 = 2$; A step in the wrong direction?", FQXi essay contest (2012), http://www.fqxi.org/community/forum/topic/1449

[2] G. M. Dixon, "Division Algebras, Lattices, Physics, Windmill Tilting", CreateSpace (2011).