Beyond complex number exponentiation: Two prototypes.

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Research statement

- Restate defining relations for Complex Numbers in different ways, then
- modify exactly one of these relations to see how it can be made to work.

Context

- **Charles A. Muses** envisioned certain number systems in 1970s.
- Definitions were not proper, but somehow intriguing.
- John Shuster and I investigated two of these, then we "fixed" them.

“Fixed them”?

“Fixing” a number system?

- Capture the essence.
- Keep as similar as possible to the Complex Numbers.
- “**W Space**” after “w number”,
- “**PQ Space**” after “pq number”.

Part 1

"W Space": Elliptic complex numbers with dual multiplication

Restating the complexes

$\mathbb{C}$ is 2D vector sspace.

$\{i, -i\}$ is solution set to:

$$\begin{aligned} \mathrm{conj}\,(z) + z &= 0, \\ \mathrm{conj}\,(z) \cdot z &= 1 = z \cdot \mathrm{conj}\,(z)\,. \end{aligned}$$

W space $\mathbb{W}$ is also over the real 2D plane.
$\{(w),(-w)\}$ is solution set to:

$$\begin{aligned}\mathrm{conj}\,(z)+z &= 1,\\ \mathrm{conj}\,(z)\cdot z &= 1 = z\cdot\mathrm{conj}\,(z)\,.\end{aligned}$$

Addition?

- Simply vector space addition.

Multiplication? From

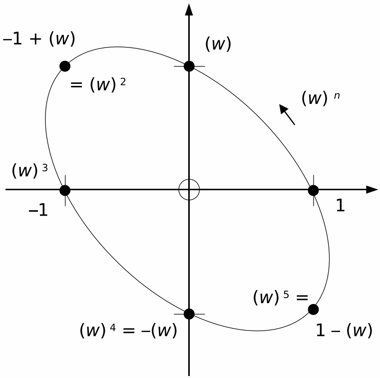
$$\begin{aligned} \mathrm{conj}\,(z) + z &= 1, \\ \mathrm{conj}\,(z) \cdot z &= 1 = z \cdot \mathrm{conj}\,(z)\,, \end{aligned}$$

follows:

$$\begin{aligned} \mathrm{conj}\,((w)) &= 1 - (w) \\ &\Rightarrow (w)^2 = (w) - 1, \\ \mathrm{conj}\,((-w)) &= 1 - (-w) \\ &\Rightarrow (-w)^2 = (-w) - 1. \end{aligned}$$

Two distinct points in the plane.

Figure:



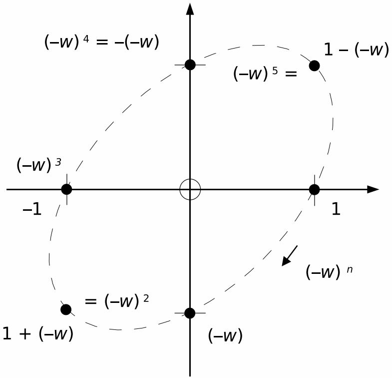
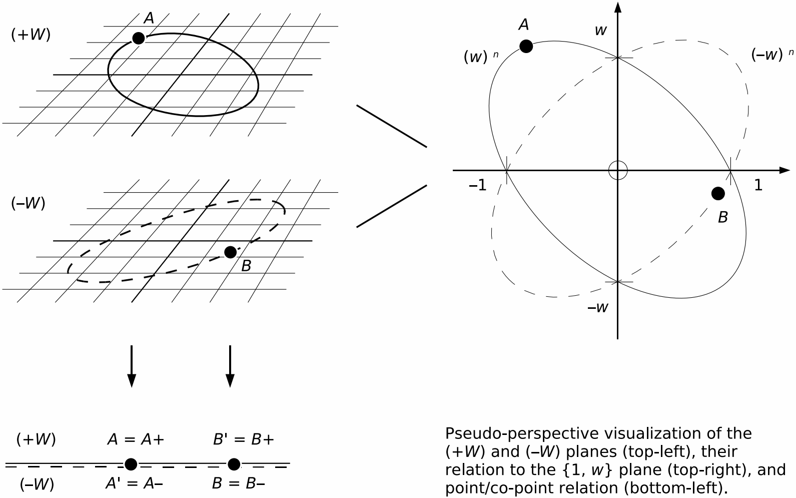


Figure:



Pseudo-perspective visualization of the (+W) and (–W) planes (top-left), their relation to the {1, w} plane (top-right), and point/co-point relation (bottom-left).

W Space $\mathbb{W}$ contains two subspaces:

- $+\mathbb{W}$ with $\langle +\mathbb{W}, \times, + \rangle$,
- $-\mathbb{W}$ with $\langle -\mathbb{W}, \circ, + \rangle$.
- Each **isomorphic** to $\mathbb{C}$, but
- different multiplication.

Side note: 2-category

On a side note:

$$\langle +\mathbb{W}, \times, + \rangle = \langle \mathbb{C}, \cdot, + \rangle = \langle -\mathbb{W}, \circ, + \rangle \,.$$

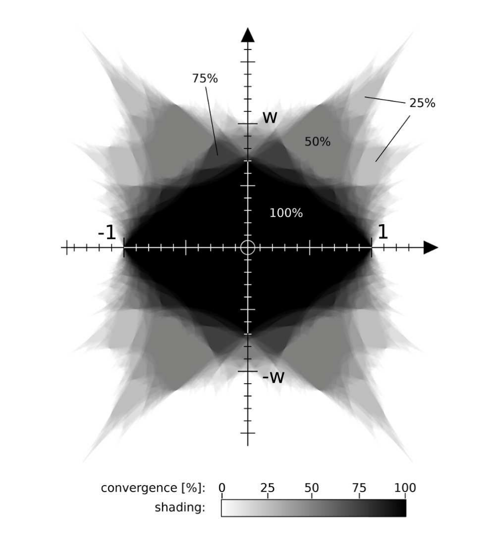
Define functor f:

- $f(\times) = \circ$, $f(\circ) = \times$.

$\Longrightarrow$ $\mathbb{W}$ is a primitive 2-category.

"Multi-Star"

Repeated squaring:
(1) Start with coordinate,
(2) square in $+\mathbb{W}$, $-\mathbb{W}$,
(3) keep both results,
(4) repeat "infinitely",
(5) plot convergence %.



Exp(Z) from Taylor polynomial

$\exp(Z)$ fractal:

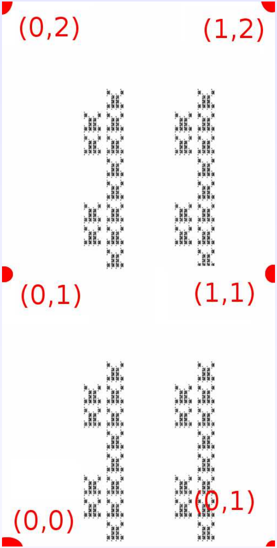
(1) $1 + z + \frac{z^2}{2!} + \ldots + \frac{z^n}{n!}$

(2) $z^n = \left\{ \underbrace{z \cdot z \cdot \ldots \cdot z}_{n \text{ times}} \right\}$

(3) $\cdot$ is either $\times$ or $\circ$

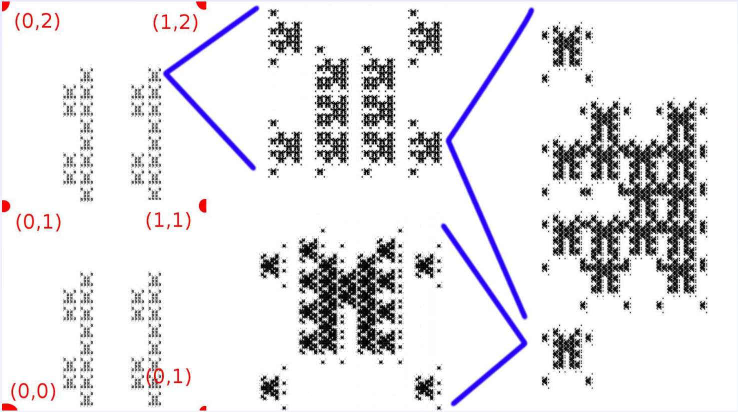
(4) keep *all* results,

(5) repeat "infinitely".



$\exp(w)$

Fractal zoom-in



Part 2

"PQ Space": Doubly nilpotent numbers in the 2D plane

$\mathbb{C}$ has additive group $\mathbb{C}^+$ from 2D vector space addition (radius r, angle t).

Multiplicative group $\mathbb{C}^\times$ with norm s:

$$s(A \cdot B) = s(A)\, s(B)\,.$$

Multiplication adds angles t:

$$t(A \cdot B) = t(A) + t(B)\,.$$

Sets $\mathbb{C}^+$ and $\mathbb{C}^\times$ are identical, $r = s$.

$\mathbb{PQ}$ has additive group $\mathbb{PQ}^{+}$ from 2D vector space addition (radius r, angle t).

Multiplicative group $\mathbb{PQ}^{\times}$ with norm s:

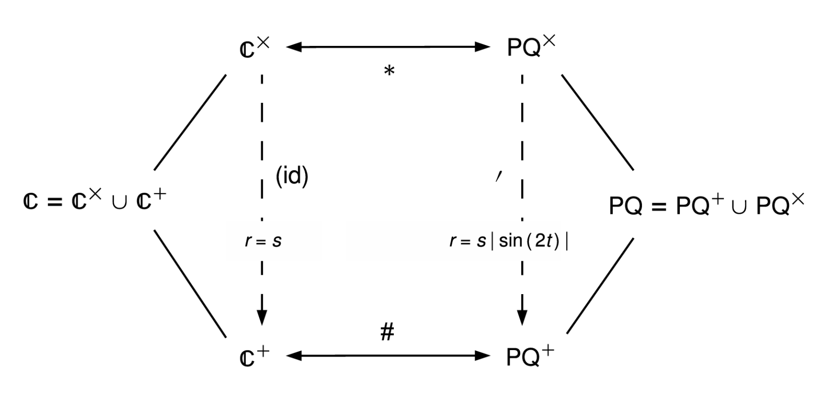
$$s\left(A\cdot B\right)=s\left(A\right)s\left(B\right).$$

Multiplication adds angles t:

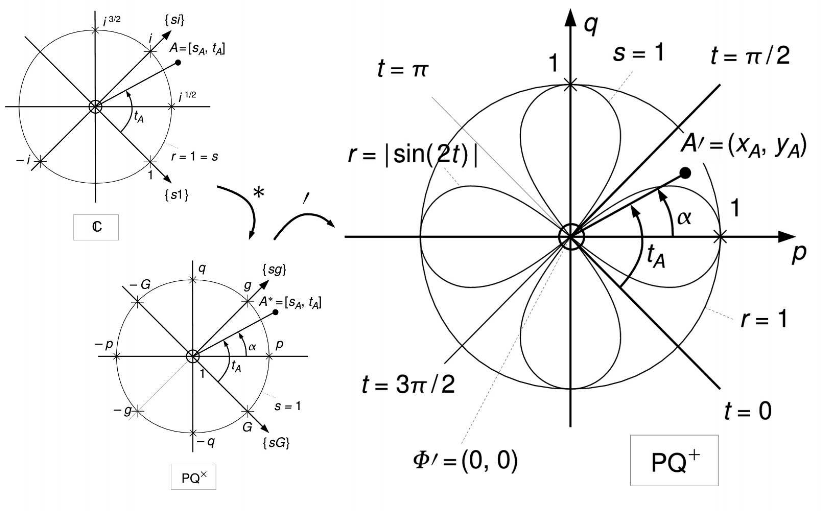
$$t\left(A\cdot B\right)=t\left(A\right)+t\left(B\right).$$

Set $\mathbb{PQ}^{\times}$ maps into $\mathbb{PQ}^{+}$ as $r=s\left|\sin\left(2t\right)\right|$.

Mappings between additive and multiplicative groups



Mappings between additive and multiplicative groups



Algebraic properties of PQ space

"Doubly nilpotent" because the square of both axes map into the additive identity:

$$p^2 \overset{\prime}{\mapsto} (0,0)$$
$$q^2 \overset{\prime}{\mapsto} (0,0)$$

Nondistributive because for $a, b, c, d \in \mathbb{R}$:

$$(ap + bq) \times (cp + dq) \neq$$
$$acp^2 + abcd\,(p \times q) + bdq^2 \overset{\prime}{\mapsto} (0,0)\,.$$

Mapping a multiplicative subspace into an additive subspace,

$$\mathbb{PQ}^{\times} \xrightarrow{\prime} \mathbb{PQ}^{+},$$

may give rise to "nondistributive algebra" class.

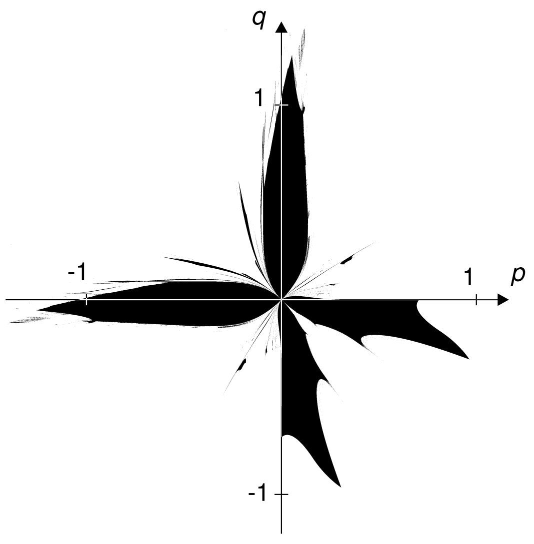
Can PQ Space be written as a 2-category, e.g. with functor

$$f\left(\mathbb{PQ}^{\times}\right) = \mathbb{PQ}^{+},$$
$$f\left(\mathbb{PQ}^{+}\right) = \mathbb{PQ}^{\times}?$$

Not sure.

"Papillon"

Mandelbrot algorithm:
(1) Start at $(0, 0)$,
(2) add (a, b),
(3) square,
(4) repeat at (2).
The final set are all nondivergent (a, b).



Outlook

"W Space" and "PQ Space" are fun.
Both have real exponentials.
Generalized exponentiation in 2D?
"Applied 2-categories"?
Nondistributive algebra?
Burgin hypernumbers?

Denver "3rd milehigh" in August!

Thank you for your attention!

References

[1] J. A. Shuster, J. Köplinger, Elliptic complex numbers with dual multiplication, Appl. Math. Comput. 216 (2010), pp. 3497-3514. personal version http://www.jenskoeplinger.com/P/PaperShusterKoepl_WSpace.pdf

[2] J. A. Shuster, J. Köplinger, Doubly nilpotent numbers in the 2D plane, Appl. Math. Comput. 217 (2011), pp. 7295-7310. personal version: http://www.jenskoeplinger.com/P/PaperShusterKoepl-PQSpace.pdf