GENERALIZED GAMMA DISTRIBUTION AS EIGENFUNCTION

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Abstract. The gamma distribution is defined over the reals, with certain restrictions on its parameters when used to describe valid random variable distributions in statistics. By postulate, the work here aims at weakening these parameter restrictions, and allowing certain higher-dimensional algebras beyond the reals as well. The goal is to find linear differential operators to which such generalized gamma distribution is an eigenfunction with real eigenvalue. Motivation comes from physics, where the structure of such equation, to be found, may have similarity to the Dirac equation with electromagnetic field.

Contents

1. Background

1.1. **The gamma distribution as eigenfunction.** The gamma distribution Γ has a probability density function over $x \in \mathbb{R}$, $x \geq 0$, with the following general structure (given some constants $a, b, c \in \mathbb{R}$ with certain constraints):

$$\Gamma(x; a, b, c) = a x^b e^{cx}. \tag{1.1}$$

With

$$\frac{d}{dx} x^b = \frac{d}{dx} \exp((\ln x) b) = \frac{d}{dx}((\ln x) b) \exp((\ln x) b) = \frac{b}{x} \exp((\ln x) b) = \frac{b}{x} x^b, \tag{1.2}$$

and

$$\frac{d}{dx} e^{cx} = c e^{cx}, \tag{1.3}$$

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the differential of the gamma distribution in the reals is:

$$\frac{d}{dx}\Gamma(x;a,b,c) = a\left(\frac{d}{dx}x^b\right)e^{cx} + ax^b\left(\frac{d}{dx}e^{cx}\right) = \left(\frac{b}{x}+c\right)\Gamma(x;a,b,c). \tag{1.4}$$

That is, the gamma distribution is - within the constraints on a, b, c - eigenfunction to an operator $D_{\mathbb{R}} := \left(\frac{d}{dx} - \frac{b}{x}\right)$ with eigenvalue c:

$$D_{\mathbb{R}} := \left(\frac{d}{dx} - \frac{b}{x}\right), \qquad D_{\mathbb{R}}\Gamma(x;a,b,c) = c\,\Gamma(x;a,b,c). \tag{1.5}$$

While the gamma distribution is defined over the reals, this still holds when allowing x and a, b, c to be complex-valued, $x, a, b, c \in \mathbb{C}$.

1.2. **Dirac equation of a free spin-1/2 particle.** The Dirac equation in physics is the quantum equation of motion of a free spin-1/2 particle. It is an operator-eigenfunction relation that is typically expressed in four complex dimensions, where the operator D takes the shape of a 4×4 complex-valued matrix. With some notational elegance removed, the often used "Dirac representation" of the Dirac equation of a free spin-1/2 particle with mass $m \in \mathbb{R}$, $m > 0$, can be written in four-dimensional spacetime $x = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4$ with complex-valued wave functions $\Psi(x)$, $\Psi : \mathbb{R}^4 \to \mathbb{C}^4$, $\Psi(x) = (\psi_0(x), \psi_1(x), \psi_2(x), \psi_3(x))$ where $\psi_\mu : \mathbb{R}^4 \to \mathbb{C}$, $\mu \in \{0, 1, 2, 3\}$, spacetime derivatives $\partial_\mu := \frac{\partial}{\partial_\mu}$, as:

$$D(\partial_\mu)_{\text{Dirac}} := \begin{pmatrix} i\partial_0 & 0 & -i\partial_3 & -i\partial_1 - \partial_2 \\ 0 & i\partial_0 & -i\partial_1 + \partial_2 & i\partial_3 \\ i\partial_3 & i\partial_1 + \partial_2 & -i\partial_0 & 0 \\ i\partial_1 - \partial_2 & -i\partial_3 & 0 & -i\partial_0 \end{pmatrix}, \tag{1.6}$$

$$D(\partial_\mu)_{\text{Dirac}}\Psi(x) = m\Psi(x). \tag{1.7}$$

Typically, all calculations are done in the complex numbers.

In preparation for the work here, all real- and complex-valued components of equation (1.7) are now written explicitly, writing $\psi_\mu(x) := \left(\psi_\mu^{\mathrm{r}}(x), \psi_\mu^{\mathrm{i}}(x)\right)$ for the real and imaginary parts of the wave functions, respectively:

$$i\partial_0\left(\psi_0^{\mathrm{r}} + i\psi_0^{\mathrm{i}}\right) - i\partial_3\left(\psi_2^{\mathrm{r}} + i\psi_2^{\mathrm{i}}\right) - (i\partial_1 + \partial_2)\left(\psi_3^{\mathrm{r}} + i\psi_3^{\mathrm{i}}\right) = m\left(\psi_0^{\mathrm{r}} + i\psi_0^{\mathrm{i}}\right),$$

$$-\partial_0\psi_0^{\mathrm{i}} + \partial_1\psi_3^{\mathrm{i}} - \partial_2\psi_3^{\mathrm{r}} + \partial_3\psi_2^{\mathrm{i}} = m\psi_0^{\mathrm{r}}, \tag{1.8}$$

$$+\partial_0\psi_0^{\mathrm{r}} - \partial_1\psi_3^{\mathrm{r}} - \partial_2\psi_3^{\mathrm{i}} - \partial_3\psi_2^{\mathrm{r}} = m\psi_0^{\mathrm{i}}, \tag{1.9}$$

$$i\partial_0\left(\psi_1^{\mathrm{r}} + i\psi_1^{\mathrm{i}}\right) + (-i\partial_1 + \partial_2)\left(\psi_2^{\mathrm{r}} + i\psi_2^{\mathrm{i}}\right) + i\partial_3\left(\psi_3^{\mathrm{r}} + i\psi_3^{\mathrm{i}}\right) = m\left(\psi_1^{\mathrm{r}} + i\psi_1^{\mathrm{i}}\right),$$

$$-\partial_0\psi_1^{\mathrm{i}} + \partial_1\psi_2^{\mathrm{i}} + \partial_2\psi_2^{\mathrm{r}} - \partial_3\psi_3^{\mathrm{i}} = m\psi_1^{\mathrm{r}}, \tag{1.10}$$

$$\partial_0\psi_1^{\mathrm{r}} - \partial_1\psi_2^{\mathrm{r}} + \partial_2\psi_2^{\mathrm{i}} + \partial_3\psi_3^{\mathrm{r}} = m\psi_1^{\mathrm{i}}, \tag{1.11}$$

$$i\partial_3\left(\psi_0^{\mathrm{r}} + i\psi_0^{\mathrm{i}}\right) + (i\partial_1 + \partial_2)\left(\psi_1^{\mathrm{r}} + i\psi_1^{\mathrm{i}}\right) - i\partial_0\left(\psi_2^{\mathrm{r}} + i\psi_2^{\mathrm{i}}\right) = m\left(\psi_2^{\mathrm{r}} + i\psi_2^{\mathrm{i}}\right),$$

$$\partial_0\psi_2^{\mathrm{i}} - \partial_1\psi_1^{\mathrm{i}} + \partial_2\psi_1^{\mathrm{r}} - \partial_3\psi_0^{\mathrm{i}} = m\psi_2^{\mathrm{r}}, \tag{1.12}$$

$$-\partial_0\psi_2^{\mathrm{r}} + \partial_1\psi_1^{\mathrm{r}} + \partial_2\psi_1^{\mathrm{i}} + \partial_3\psi_0^{\mathrm{r}} = m\psi_2^{\mathrm{i}}, \tag{1.13}$$

$$(i\partial_1 - \partial_2)\left(\psi_0^{\mathrm{r}} + i\psi_0^{\mathrm{i}}\right) - i\partial_3\left(\psi_1^{\mathrm{r}} + i\psi_1^{\mathrm{i}}\right) - i\partial_0\left(\psi_3^{\mathrm{r}} + i\psi_3^{\mathrm{i}}\right) = m\left(\psi_3^{\mathrm{r}} + i\psi_3^{\mathrm{i}}\right),$$

$$\partial_0\psi_3^{\mathrm{i}} - \partial_1\psi_0^{\mathrm{i}} - \partial_2\psi_0^{\mathrm{r}} + \partial_3\psi_1^{\mathrm{i}} = m\psi_3^{\mathrm{r}}, \tag{1.14}$$

$$-\partial_0\psi_3^{\mathrm{r}} + \partial_1\psi_0^{\mathrm{r}} - \partial_2\psi_0^{\mathrm{i}} - \partial_3\psi_1^{\mathrm{r}} = m\psi_3^{\mathrm{i}}. \tag{1.15}$$

1.3. **Dirac equation with electromagnetic field.** In physics, an electromagnetic field $A(x)$, $A : \mathbb{R}^4 \to \mathbb{R}^4$ acting on that particle is introduced by asking what shape a modified operator $D(\partial_\mu)_{\mathrm{Dirac}} \mapsto D(\partial_\mu; A(x))_{\mathrm{Dirac}}$ would take for this eigenvalue equation to still hold, if one transforms the wave function $\Psi(x) \mapsto \Psi(x; A(x))$ by a phase $e^{i\theta}$, $\theta \in \mathbb{R}$:

$$\Psi(x; A(x)) := e^{i\theta}\Psi(x) \qquad \text{with } A_\mu := \frac{\partial\theta}{\partial x_\mu}, \tag{1.16}$$

$$D(\partial_\mu; A(x))_{\mathrm{Dirac}} := D(\partial_\mu + iA_\mu(x))_{\mathrm{Dirac}}, \tag{1.17}$$

$$D(\partial_\mu; A(x))_{\mathrm{Dirac}}\Psi(x; A(x)) = m\Psi(x; A(x)). \tag{1.18}$$

That is, the operator $D(\partial_\mu; A(x))_{\mathrm{Dirac}}$ to which the transformed wave function $\Psi(x; A(x))$ is eigenfunction with eigenvalue m, is obtained by adding a term $iA_\mu(x)$ to each occurence of the corresponding ∂_μ term in the free operator $D(\partial_\mu)_{\mathrm{Dirac}}$, to become $D(\partial_\mu + iA_\mu(x))_{\mathrm{Dirac}}$.

1.4. **A primitive 1D version of the Dirac equation with EM field.** The Dirac equation with electromagnetic field (1.18) uses four-dimensional complex wave functions $\Psi(x)$ that take four real spacetime as parameters, $\Psi : \mathbb{R}^4 \to \mathbb{C}^4$. By supposition, a primitive one-dimensional version of the same equation, with $D(\partial_\mu)_{\mathrm{Dirac}} \mapsto \frac{d}{dx}$, $x \in \mathbb{R}$, $A : \mathbb{R} \to \mathbb{R}$, $\Psi : \mathbb{R} \to \mathbb{C}$, could take the form:

$$\left(i\frac{d}{dx} - A(x)\right)\Psi = m\Psi. \tag{1.19}$$

Assuming a potential $A(x) \sim 1/x$ for this degenerate "electric" field, i.e., with the same distance behavior as a conventional point charge would have in three dimensions, this primitive one-dimensional version of the Dirac equation would have the structure of the eigenvalue equation on the complexified gamma distribution above (equation 1.5).

1.5. **Dirac equation as split-octonion product.** Split-octonions $\mathbb{O}'$ are eight dimensional algebras over the reals, $\mathbb{O}' := \left\langle \mathbb{R}^8, *, + \right\rangle$ with a generally nonassociative product [WP:SplitOctonion]. They are closely related to octonions $\mathbb{O}$, the only normed division algebras in eight real dimensions [WP:Octonion]. While split-octonions have subspaces where multiplication is not always invertible [BentzDray2018], they are the unique algebras in eight real dimensions, next to the octonions, that are still composition algebras [Jacobson1958, WP:CompositionAlgebra, PM:CompositionAlgebra].

Just like representations of the Dirac equation are not unique, so are representations of the split-octonion algebra. A representation can be chosen with basis $b_{\mathbb{O}'} = \{e_\nu\} := \{1, i_1, i_2, i_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7\}$, $\nu \in \{0, \ldots 7\}$, $(i_{1,2,3})^2 = -1$, $(\varepsilon_{4,5,6,7})^2 = +1$, and associative triplets $\{(e_\rho, e_\sigma, e_\tau)\}$ as $\{\rho\sigma\tau\} := \{123, 145, 176, 246, 257, 347, 365\}$. For convenience, a complete multiplication table for the pairwise product of all basis elements is given in table 1.

With this, it is possible to express the Dirac equation in Dirac representation (equation 1.7) as a simple split-octonion product [Koeplinger2006]:

$$D(\partial_\mu)_{\mathbb{O}'} := (0, \partial_0, 0, 0, \quad 0, -\partial_3, \partial_2, -\partial_1), \tag{1.20}$$

$$\Psi_{\mathbb{O}'}(x) := \left(\psi_0^{\mathrm{r}}(x), \psi_0^{\mathrm{i}}(x), \psi_1^{\mathrm{r}}(x), \psi_1^{\mathrm{i}}(x), \quad \psi_2^{\mathrm{r}}(x), -\psi_2^{\mathrm{i}}(x), -\psi_3^{\mathrm{r}}(x), -\psi_3^{\mathrm{i}}(x)\right), \tag{1.21}$$

$$D(\partial_\mu)_{\mathbb{O}'}\Psi_{\mathbb{O}'}(x) = m\Psi_{\mathbb{O}'}(x). \tag{1.22}$$

$\mathbb{O}'$	$\mathbf{1}$	$\boldsymbol{i_1}$	$\boldsymbol{i_2}$	$\boldsymbol{i_3}$	$\boldsymbol{\varepsilon_4}$	$\boldsymbol{\varepsilon_5}$	$\boldsymbol{\varepsilon_6}$	$\boldsymbol{\varepsilon_7}$
$\mathbf{1}$	1	i_1	i_2	i_3	ε_4	ε_5	ε_6	ε_7
$\boldsymbol{i_1}$	i_1	-1	i_3	$-i_2$	ε_5	$-\varepsilon_4$	$-\varepsilon_7$	ε_6
$\boldsymbol{i_2}$	i_2	$-i_3$	-1	i_1	ε_6	ε_7	$-\varepsilon_4$	$-\varepsilon_5$
$\boldsymbol{i_3}$	i_3	i_2	$-i_1$	-1	ε_7	$-\varepsilon_6$	ε_5	$-\varepsilon_4$
$\boldsymbol{\varepsilon_4}$	ε_4	$-\varepsilon_5$	$-\varepsilon_6$	$-\varepsilon_7$	1	$-i_1$	$-i_2$	$-i_3$
$\boldsymbol{\varepsilon_5}$	ε_5	ε_4	$-\varepsilon_7$	ε_6	i_1	1	i_3	$-i_2$
$\boldsymbol{\varepsilon_6}$	ε_6	ε_7	ε_4	$-\varepsilon_5$	i_2	$-i_3$	1	i_1
$\boldsymbol{\varepsilon_7}$	ε_7	$-\varepsilon_6$	ε_5	ε_4	i_3	i_2	$-i_1$	1

TABLE 1. Multiplication table (row × column) for basis elements of the chosen split-octonion $\mathbb{O}'$ representation.

You can verify that this is indeed a representation of the Dirac equation by direct comparison of the eight real components of equation (1.22). With

$$\begin{aligned} D\left(\partial_\mu\right)_{\mathbb{O}'}\Psi_{\mathbb{O}'}(x) &= (0,\partial_0,0,0,\quad 0,-\partial_3,\partial_2,-\partial_1)\left(\psi_0^{\mathrm{r}},\psi_0^{\mathrm{i}},\psi_1^{\mathrm{r}},\psi_1^{\mathrm{i}},\quad \psi_2^{\mathrm{r}},-\psi_2^{\mathrm{i}},-\psi_3^{\mathrm{r}},-\psi_3^{\mathrm{i}}\right) \\ &= (-\partial_0\psi_0^{\mathrm{i}},\ \partial_0\psi_0^{\mathrm{r}},\ -\partial_0\psi_1^{\mathrm{i}},\ \partial_0\psi_1^{\mathrm{r}},\ \partial_0\psi_2^{\mathrm{i}},\ \partial_0\psi_2^{\mathrm{r}},\ -\partial_0\psi_3^{\mathrm{i}},\ \partial_0\psi_3^{\mathrm{r}}) \\ &\quad +(\partial_3\psi_2^{\mathrm{i}},\ -\partial_3\psi_2^{\mathrm{r}},\ -\partial_3\psi_3^{\mathrm{i}},\ \partial_3\psi_3^{\mathrm{r}},\ -\partial_3\psi_0^{\mathrm{i}},\ -\partial_3\psi_0^{\mathrm{r}},\ -\partial_3\psi_1^{\mathrm{i}},\ \partial_3\psi_1^{\mathrm{r}}) \\ &\quad +(-\partial_2\psi_3^{\mathrm{r}},\ -\partial_2\psi_3^{\mathrm{i}},\ \partial_2\psi_2^{\mathrm{r}},\ \partial_2\psi_2^{\mathrm{i}},\ \partial_2\psi_1^{\mathrm{r}},\ -\partial_2\psi_1^{\mathrm{i}},\ \partial_2\psi_0^{\mathrm{r}},\ \partial_2\psi_0^{\mathrm{i}}) \\ &\quad +(\partial_1\psi_3^{\mathrm{i}},\ -\partial_1\psi_3^{\mathrm{r}},\ \partial_1\psi_2^{\mathrm{i}},\ -\partial_1\psi_2^{\mathrm{r}},\ -\partial_1\psi_1^{\mathrm{i}},\ -\partial_1\psi_1^{\mathrm{r}},\ \partial_1\psi_0^{\mathrm{i}},\ -\partial_1\psi_0^{\mathrm{r}}) \end{aligned} \tag{1.23}$$

the eight real components equation (1.22) are:

$$\begin{pmatrix} m\psi_0^{\mathrm{r}} \\ m\psi_0^{\mathrm{i}} \\ m\psi_1^{\mathrm{r}} \\ m\psi_1^{\mathrm{i}} \\ m\psi_2^{\mathrm{r}} \\ -m\psi_2^{\mathrm{i}} \\ -m\psi_3^{\mathrm{r}} \\ -m\psi_3^{\mathrm{i}} \end{pmatrix} = \begin{pmatrix} -\partial_0\psi_0^{\mathrm{i}}+\partial_3\psi_2^{\mathrm{i}}-\partial_2\psi_3^{\mathrm{r}}+\partial_1\psi_3^{\mathrm{i}} \\ \partial_0\psi_0^{\mathrm{r}}-\partial_3\psi_2^{\mathrm{r}}-\partial_2\psi_3^{\mathrm{i}}-\partial_1\psi_3^{\mathrm{r}} \\ -\partial_0\psi_1^{\mathrm{i}}-\partial_3\psi_3^{\mathrm{i}}+\partial_2\psi_2^{\mathrm{r}}+\partial_1\psi_2^{\mathrm{i}} \\ \partial_0\psi_1^{\mathrm{r}}+\partial_3\psi_3^{\mathrm{r}}+\partial_2\psi_2^{\mathrm{i}}-\partial_1\psi_2^{\mathrm{r}} \\ \partial_0\psi_2^{\mathrm{i}}-\partial_3\psi_0^{\mathrm{i}}+\partial_2\psi_1^{\mathrm{r}}-\partial_1\psi_1^{\mathrm{i}} \\ \partial_0\psi_2^{\mathrm{r}}-\partial_3\psi_0^{\mathrm{r}}-\partial_2\psi_1^{\mathrm{i}}-\partial_1\psi_1^{\mathrm{r}} \\ -\partial_0\psi_3^{\mathrm{i}}-\partial_3\psi_1^{\mathrm{i}}+\partial_2\psi_0^{\mathrm{r}}+\partial_1\psi_0^{\mathrm{i}} \\ \partial_0\psi_3^{\mathrm{r}}+\partial_3\psi_1^{\mathrm{r}}+\partial_2\psi_0^{\mathrm{i}}-\partial_1\psi_0^{\mathrm{r}} \end{pmatrix}. \tag{1.24}$$

These are exactly the eight components of conventional Dirac equation, (1.8) through (1.15) above.

It is possible to introduce an electromagnetic field by complexifying the algebra, i.e., by using coefficients instead of real ones [Koeplinger2007]. Writing i as the imaginary unit in the coefficients, and $A(x)$, $A:\mathbb{R}^4\to\mathbb{R}^4$ for the electromagnetic field as before, this takes the shape

$$D\left(\partial_\mu+\mathrm{i}A_\mu(x)\right)_{\mathbb{O}'}\Psi_{\mathbb{O}'}(x) = m\Psi_{\mathbb{O}'}(x). \tag{1.25}$$

However, not only are representations of the split-octonion algebra not unique, the algebra doesn't have to be split-octonion at all for this to work: Equation 1.22 does not use any of the basis elements i_2, i_3, or ε_4 on the left side of the product. Table 2 shows only the basis element pairs that are actually used as factors to represent the free Dirac equation.

(?)	$\boldsymbol{1}$	$\boldsymbol{i_1}$	$\boldsymbol{i_2}$	$\boldsymbol{i_3}$	$\boldsymbol{\varepsilon_4}$	$\boldsymbol{\varepsilon_5}$	$\boldsymbol{\varepsilon_6}$	$\boldsymbol{\varepsilon_7}$
$\boldsymbol{1}$	1	i_1	i_2	i_3	ε_4	ε_5	ε_6	ε_7
$\boldsymbol{i_1}$	i_1	-1	i_3	$-i_2$	ε_5	$-\varepsilon_4$	$-\varepsilon_7$	ε_6
$\boldsymbol{i_2}$								
$\boldsymbol{i_3}$								
$\boldsymbol{\varepsilon_4}$								
$\boldsymbol{\varepsilon_5}$	ε_5	ε_4	$-\varepsilon_7$	ε_6	i_1	1	i_3	$-i_2$
$\boldsymbol{\varepsilon_6}$	ε_6	ε_7	ε_4	$-\varepsilon_5$	i_2	$-i_3$	1	i_1
$\boldsymbol{\varepsilon_7}$	ε_7	$-\varepsilon_6$	ε_5	ε_4	i_3	i_2	$-i_1$	1

TABLE 2. The subset of the split-octonion multiplication table 1 that is actually used in representing the free Dirac equation (1.22).

2. Problem statement

The Dirac equation with electromagnetic field in physics, equation 1.18,

$$D\left(\partial_\mu + iA_\mu\right)_{\text{Dirac}} \Psi \quad = \quad m\Psi, \tag{2.1}$$

is a linear differential operator D acting on eigenfunctions Ψ with real eigenvalue m. Electromagnetic fields A generally have $1/x$ characteristic, i.e., they weaken in proportion of the relative distance to a point charge. Keeping the structure of this equation, a primitive one dimensional case would become equation 1.19:

$$\left(i\frac{d}{dx} - A\right)\Psi = m\Psi. \tag{2.2}$$

Schematically, this has the same structure as the eigenvalue equation of the complexified gamma distribution, equation 1.5

$$\left(\frac{d}{dx} - \frac{b}{x}\right)\Gamma = c\Gamma. \tag{2.3}$$

Using split-octonion representation with complex coefficients, the Dirac equation with electromagnetic field can be expressed as the simple product (equation 1.25):

$$D\left(\partial_\mu + \mathrm{i}A_\mu\right)_{\mathbb{O}'} \Psi \quad = \quad m\Psi. \tag{2.4}$$

While complexified split-octonions are sufficient to represent the Dirac equation this way, there are many freedoms that would allow for other algebras as well (table 2).

The problem to be solved here is to see whether some higher-dimensional algebra generalization of the conventional gamma distribution exists, such that this generalized "distribution" would be eigenfunction to a linear differential equation with structure

$$D\left(\partial_\mu + B_\mu\right)_? \Gamma = c\Gamma, \tag{2.5}$$

where the B_μ correspond to electromagnetic fields in physics.

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