INTRODUCING BURGIN HYPERNUMBERS IN ONE PAGE

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Following [1, 2], a sequence $\mathbf{a}$ of numbers a_i is written as $\mathbf{a} = (a_i)_{i\in\omega}$ where ω is the totally ordered set $\{1, 2, 3, \ldots\}$. Two sequences $\mathbf{a}$ and $\mathbf{b}$ are considered equivalent if they approximate one another from pairwise comparison of sequence members of the same index:

$$\mathbf{a} \approx \mathbf{b} \quad \longleftrightarrow \quad \lim_{i\to\infty} |a_i - b_i| = 0. \tag{0.1}$$

This equivalence relation requires a suitable definition of $|a_i - b_i|$ in the chosen space of the a_i and b_i. In this paper, only sequences over the reals will be handled, $a_i, b_i \in \mathbb{R}$. A Burgin hypernumber $\mathbf{A} = \mathrm{Hn}\,(a_i)_{i\in\omega}$ is represented by a sequence $\mathbf{a} = (a_i)_{i\in\omega}$ such that any two hypernumbers are equal exactly when the representing sequences are equivalent:

$$\begin{aligned} \mathbf{A} &= \mathrm{Hn}\,(a_i)_{i\in\omega} = \mathrm{Hn}\,(\mathbf{a})\,, \\ \mathbf{B} &= \mathrm{Hn}\,(b_i)_{i\in\omega} = \mathrm{Hn}\,(\mathbf{b})\,, \\ \mathbf{A} = \mathbf{B} &\longleftrightarrow \mathbf{a} \approx \mathbf{b}. \end{aligned}$$

In general, a Burgin hypernumber can be represented by infinitely many equivalent sequences.

In the case of sequences that approximate a real number $r \in \mathbb{R}$, Burgin hypernumbers reduce to the real numbers without exception. For example, the real number 1 could be represented by sequences $\{1, 1, 1, \ldots, 1, \ldots\}$ or $\{0, 0.9, 0.99, 0.999, \ldots, 0.999\ldots9, \ldots\}$ likewise. Burgin hypernumbers do not introduce new kinds of infinitesimals as in nonstandard analysis. Instead, they extend the reals through numbers that are represented by nonconvergent sequences. These sequences could go against infinity, or oscillate, or a combination of these two cases. For example, the following hypernumbers $\mathbf{D}$ and $\mathbf{E}$ are the same since they are represented by equivalent sequences:

$$\begin{aligned} \mathbf{d} &: \omega \to d_i = \{1, 0, 1, 0, \ldots 1, 0, \ldots\}\,, \\ \mathbf{e} &: \omega \to e_i = \left\{1, \frac{1}{2}, 1, \frac{1}{4}, \ldots 1, \frac{1}{i}, \ldots\right\}, \\ \mathbf{D} &:= \mathrm{Hn}\,(\mathbf{d})\,, \\ \mathbf{E} &:= \mathrm{Hn}\,(\mathbf{e})\,. \\ \Longrightarrow \quad \mathbf{D} &= \mathbf{E}. \end{aligned} \tag{0.2}$$

The spectrum $\mathrm{Spec}\,(\mathbf{A})$ of a hypernumber $\mathbf{A} = \mathrm{Hn}\,(\mathbf{a})$ is defined as the set of all real numbers $r \in \mathbb{R}$ that can be approximated by some subsequence $\mathbf{b}$ of $\mathbf{a}$:

$$\mathrm{Spec}\,(\mathbf{A}) \;:=\; \left\{ r \in \mathbb{R}; r = \lim_{i\to\infty} b_i \text{ for some subsequence } \mathbf{b} = (b_i)_{i\in\omega} \text{ of } \mathbf{a} \right\}.$$

For hypernumbers that are represented by sequences that approximate a real number, the spectrum is that real number. Any real number can be expressed as a sequence of rationals which approximate that number.

The example of equation (0.2) above has a $\mathrm{Spec}\,(\mathbf{A}) = \{0, 1\}$. Any two hypernumbers that are equal also have the same spectrum, although the converse is not true. For example, the number $\mathbf{C} = \mathrm{Hn}\,(\mathbf{c})$ with $c_i = \{0, 1, 0, 1, 0, \ldots 0, 1, \ldots\}$ has $\mathrm{Spec}\,(\mathbf{C}) = \{0, 1\}$. However, is $\mathbf{C}$ not equal to $\mathbf{A}$ or $\mathbf{B}$ from the example (0.2), as the representing sequences do not converge per equivalence relation (0.1).

References

[1] M. Burgin, Theory of hypernumbers and extrafunctions: functional spaces and differentiation. *Discrete Dyn. Nat. Soc.* **7** (2002), 201-212.
[2] M. Burgin, Hypernumbers and extrafunctions. *Springer* (2012).

Date: 5 March 2013.

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