Quantum states in one dimension from multivalued complex exponentiation

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Outline

Recap

"The square root of Bayesian inference" sketch
(slide set from 31 August 2019, rev 2020-01-06)

Complex-valued likelihood, prior and posterior distribution

Complexify the components of Bayes' theorem:

$$\Psi(\theta \mid x) = \frac{\phi(x \mid \theta)\,\Psi(\theta)}{\int_\theta \phi(x \mid \theta')\,\Psi(\theta')\,d\theta'}$$

- Allow prior $\Psi(\theta)$, posterior $\Psi(\theta \mid x)$, and likelihood $\phi(x \mid \theta)$ to be complex-valued[1] (by proposition).
- Although these are not real-valued "distributions" or "likelihood", keep conventional terminology.

[1] The referenced slide set uses the term "spinor-valued" which is synonymous in the complexes.

Interpretation for use in quantum mechanics

Propose for modeling quantum mechanics:

- Interpret the complexified prior $\Psi(\theta)$ and posterior $\Psi(\theta \mid x)$ as wave functions from quantum mechanics.
- Measurement of a quantum system $\Psi_{\mathrm{Q}}(\theta)$ corresponds to finding a likelihood $\phi_{\mathrm{E}}(x \mid \theta)$ such that the posterior $\Psi_{\mathrm{M}}(\theta \mid x)$ is real-valued:

$$\Psi_{\mathrm{M}}(\theta \mid x) = \frac{\phi_{\mathrm{E}}(x \mid \theta)\,\Psi_{\mathrm{Q}}(\theta)}{\int_\theta \phi_{\mathrm{E}}(x \mid \theta')\,\Psi_{\mathrm{Q}}(\theta')\,d\theta'}$$

- In a pointed way, this makes quantum mechanics the "square root of Bayesian inference".

Toy model in 1D: Particle and fields

Build a toy model in one dimension:

- Wave functions (posteriors) are built from complex-valued priors and likelihoods over a real parameter θ.
- The (noninformative) prior for a point particle of mass m is

$$\Psi(\theta; m) := e^{im\theta}.$$

- Fields $q_j/(\theta - \theta_{0,j})$ from n charges q_j at positions $\theta_{0,j}$ generate a likelihood

$$\phi(\theta; q, \theta_0) := \prod_{j=1}^{n} |\theta - \theta_{0,j}|^{iq_j} \equiv \phi(q, \theta_0 \mid \theta).$$

- $\Psi(\theta; m)$ and $\phi(\theta; q, \theta_0)$ quantify ignorance of complex phase.

Recovering the Born rule: "Making good choices"

The wave function for a particle with mass m under the influence of n charges q_j at positions $\theta_{0,j}$ Then is:

$$\Psi_{\mathrm{Q}}(\theta; q_j, \theta_{0,j}, m) \propto \left(\prod_{j=1}^{n} |\theta - \theta_{0,j}|^{iq_j} \right) e^{im\theta} \equiv \Psi_{\mathrm{Q}}(\theta \mid q_j, \theta_{0,j}, m) .$$

These are eigenfunctions with real eigenvalue m to operator

$$\hat{D} := -i \frac{\partial}{\partial \theta} - \sum_{j=1}^{n} \frac{q_j}{\theta - \theta_{0,j}} .$$

Requiring the wave functions to be eigenfunctions to $\hat{D}$ with real eigenvalue m is therefore consistent with "making good choices" - in the Bayesian sense - for priors and likelihoods.

Recap

A multivalued complex exponentiation
(slide set from 22 September 2019, rev 2020-01-07)

Exponential function and multivalued logarithm

Define exponential function

$$\exp x := 1 + \sum_{j=1}^{\infty} \frac{1}{j!} \left(\underbrace{x * \ldots * x}_{j \text{ times}} \right)$$

as inverse to a generally multivalued logarithm:

$$\begin{aligned} \log x &:= \{y \mid \exp y = x\} \\ &= \{\ln|x| + i\,(\Phi(x) + 2\pi k)\}\,. \end{aligned}$$

Here, $\ln|x|$ is the real-valued logarithm, and $\Phi(x)$ the phase angle from a chosen principal branch.

Define expressions x^y as generally multivalued

Understand all expressions are sets of numbers, and identify:

$$\log x^y := (\log x)\, y \qquad (x, y \in \mathbb{C} \setminus \{0\}),$$
$$x^y := \exp((\log x)\, y).$$

Multivalued identities that hold in general:

$$x_1^y x_2^y = (x_1 x_2)^y,$$
$$\frac{d}{dx} x^y = \left(\frac{y}{x}\right) x^y,$$

Identities that require the same branch in each factor ("@k"):

$$\{x^{y_1} x^{y_2}\}_{@k} = x^{y_1 + y_2},$$
$$\frac{d}{dy}(x^y) = \{(\log x)\, x^y\}_{@k}.$$

Putting it all together: Enumerating quantum states from branches in the multivalued logarithm

Multivalued exponentials and $\sqrt{\text{Bayes}}$

Allow multivalued exponentials in "the square root of Bayesian inference" sketch.

- Careful: Exponentials of the Euler number are multi-valued!
- Specifically: Particle prior from earlier,

$$\begin{aligned}\Psi(\theta; m) \overset{??}{:=} e^{im\theta} &= \exp((\log e)\, im\theta) \\ &= \{\exp((1 + i\, 2\pi k)\, im\theta)\} \\ &= \exp(im\theta)\{\exp(-2\pi km\theta)\}.\end{aligned}$$

- Interpreting the branch enumerator k as quantization of m, the $\exp(-2\pi km\theta)$ terms are divergent for $\theta \to -\infty$, $k > 0$ (and for $\theta \to \infty$, $k < 0$ likewise). This looks unphysical.

Noninformative particle prior redefined

Instead of $e^{im\theta}$, choose different particle prior:

$$\begin{aligned}\Psi(\theta; m, \varphi) &:= u(\varphi)^{\bar{m}\theta}, \\ \text{with } u(\varphi) &:= \exp(i\varphi),\ \varphi \in \mathbb{R},\ \bar{m} := \frac{m}{2\pi}.\end{aligned}$$

- All $u(\varphi)$ lie on the unit circle in $\mathbb{C}$.
- All values of $u(\varphi)^{\bar{m}\theta}$ also lie on the unit circle:

$$\begin{aligned}u(\varphi)^{\bar{m}\theta} &= \exp((\log u(\varphi))\,\bar{m}\theta) \\ &= \{\exp((i\varphi + i\,2\pi k)\,\bar{m}\theta)\} \\ &= \left\{\exp\left(i\left(\frac{\varphi}{2\pi} + k\right) m\theta\right)\right\}.\end{aligned}$$

Properties of the redefined particle prior

- The branch enumerator $k \in \mathbb{Z}$ doesn't change the geometry of the solution space,

$$\Psi(\theta; m, \varphi) = u(\varphi)^{\bar{m}\theta} = \left\{ \exp\left(i \left(\frac{\varphi}{2\pi} + k \right) m\theta \right) \right\},$$

just the rate at which $\Psi(\theta; m, \varphi)$ changes when θ varies.

- For $\varphi = \pi$ we have $u(\pi) = -1$ and

$$\Psi(\theta; m, \pi) = (-1)^{\bar{m}\theta} = \left\{ \exp\left(i \left(\frac{1}{2} + k \right) m\theta \right) \right\}.$$

- The $\left(\frac{1}{2} + k\right) m$ term resembles quantized energy $\sim m$ of sorts. From afar, looks suitable for physics.

Likelihoods generated by fields

Recalling fields $q_j/(\theta - \theta_{0,j})$ from n charges q_j at positions $\theta_{0,j}$ to generate a likelihood

$$\phi(\theta; q, \theta_0) := \prod_{j=1}^{n} |\theta - \theta_{0,j}|^{iq_j}.$$

For a given field j, multivalued exponentiation yields

$$\begin{aligned}\phi_j(\theta; q, \theta_0) &= |\theta - \theta_{0,j}|^{iq_j} = \exp((\log|\theta - \theta_{0,j}|)\, iq_j) \\ &= \{\exp((\ln|\theta - \theta_{0,j}| + i\, 2\pi k_j)\, iq_j)\} \\ &= \exp((\ln|\theta - \theta_{0,j}|)\, iq_j)\, \{\exp(-2\pi k_j q_j)\}.\end{aligned}$$

The branch enumerators k_j quantize q_j. Looks suitable for physics.

Did we make good choices? Born rule

Prior and likelihood choices are subjective, ad-hoc.

- Why make those choices and not others?
- Following interpretation from "quantum Bayesianism" ("QBism"), the Born rule identifies good choices.

$\Longrightarrow$ Test this!

Pick a sample scenario and apply the Born rule:

- Find an operator to which the multivalued posterior is a set of eigenfunctions with a real eigenvalue each.
- Examine this set of eigenfunctions for orthogonality.
- Execute measurement, discuss the result.

Test the theory end-to-end, for a specific experimental set-up

Experimental setup

The following is the experimental set-up:

- One particle is subjected to a field generated by one charge.
- The particle prior is given after chosing a mass, m, and a phase φ as:

$$\Psi(\theta; m, \varphi) := (\exp(i\varphi))^{\bar{m}\theta} = \left\{ \exp\left(i \left(\frac{\varphi}{2\pi} + k \right) m\theta \right) \right\}.$$

- A field is generated by a charge q at the origin, $\theta_0 = 0$:

$$\phi(\theta; q, \theta_0 = 0) := |\theta|^{iq} = Q \exp((\ln|\theta|) iq).$$

(with constant set $Q := \{\exp(-2\pi k_1 q)\}$, $k_1 \in \mathbb{Z}$).

Operator, eigenfunctions, eigenvalues

The wave function of the system, Ψ_{Q}, is the posterior

$$\begin{aligned}\Psi_{\mathrm{Q}}(\theta; q, m, \varphi) =& \phi(\theta; q, \theta_0 = 0)\,\Psi(\theta; m, \varphi)\\ =& |\theta|^{iq}(\exp(i\varphi))^{\bar{m}\theta}\\ =& Q\exp((\ln|\theta|)\,iq)\left\{\exp\left(i\left(\frac{\varphi}{2\pi}+k\right)m\theta\right)\right\}.\end{aligned}$$

These are eigenfunctions to operator

$$\hat{D} := -i\frac{\partial}{\partial\theta} - \frac{q}{\theta}$$

with real eigenvalues

$$\hat{D}\Psi_{\mathrm{Q}} = \left\{\left(\frac{\varphi}{2\pi}+k\right)m\,\Psi_{\mathrm{Q}}\right\}_{@k}.$$

Operator, eigenfunctions, eigenvalues: special cases

On a side note, the special cases of $\varphi \in \{0, \pi\}$ have eigenvalues that are symmetric in k around $k = 0$:

$$\Psi_{\mathrm{Q},0} = 1^{\bar{m}\theta} \, |\theta|^{iq} \quad \Longrightarrow \quad \hat{D}\Psi_{\mathrm{Q},0} = \{km\,\Psi_{\mathrm{Q},0}\}_{@k} \, .$$

$$\Psi_{\mathrm{Q},\pi} = (-1)^{\bar{m}\theta} \, |\theta|^{iq} \quad \Longrightarrow \quad \hat{D}\Psi_{\mathrm{Q},\pi} = \left\{ \left(\frac{1}{2} + k \right) m\,\Psi_{\mathrm{Q},\pi} \right\}_{@k}$$

Suggestive as these may look, without a time component the km and $\left(\frac{1}{2} + k\right) m$ can't really be interpreted as "energy" eigenvalues. Instead, the Ψ may be some kind of particle/field building blocks.

Orthogonality of eigenfunctions

The eigenfunctions in

$$\Psi_{\mathrm{Q}}(\theta; q, m, \varphi) = Q \exp((\ln|\theta|) iq) \left\{ \exp\left(i \left(\frac{\varphi}{2\pi} + k \right) m\theta \right) \right\}$$

are orthogonal: For any pair $k_a, k_b \in \mathbb{Z}$ there is

$$\begin{aligned} &\int_{-\infty}^{\infty} \Psi_{\mathrm{Q}}(\theta'; q, m, \varphi)_{@k_a} \Psi_{\mathrm{Q}}^*(\theta'; q, m, \varphi)_{@k_b} \, d\theta' \\ =& Q^2 \int_{-\infty}^{\infty} \exp(i(k_a - k_b) m\theta') \, d\theta' \\ =& \frac{Q^2}{m} \delta(k_a - k_b) \end{aligned}$$

the Dirac-δ function with a constant set $Q^2/m = Q \times Q/m$.

Measurement: Likelihood from experimental setup

"Measurement" is the set-up of an experiment that is described exactly by a likelihood function that yields a real-valued posterior.

- Prior Ψ_Q is the entire wave function:

$$\Psi_\text{Q}(\theta; q, m, \varphi) = Q \exp((\ln|\theta|)\, iq) \left\{\exp\left(i\left(\frac{\varphi}{2\pi} + k\right) m\theta\right)\right\}.$$

- Likelihood ϕ_E is effected by the experiment:

$$\phi_\text{E}(\theta; q, m, \varphi) := \exp(-(\ln|\theta|)\, iq) \left\{\exp\left(-i\left(\frac{\varphi}{2\pi} + k\right) m\theta\right)\right\}.$$

- The product of Ψ_Q and ϕ_E is constant in θ, and real for the same branches k:

$$\{\phi_\text{E}\Psi_\text{Q}\}_{@k} = Q = \{\exp(-2\pi k_1 q)\} \in \mathbb{R}.$$

Measurement: Probability pseudo-distribution

In this experimental set-up, we found an operator

$$\hat{D} := -i\frac{\partial}{\partial\theta} - \frac{q}{\theta}$$

to yield real eigenvalues

$$\hat{D}\Psi_{\mathrm{Q}} = \left\{\left(\frac{\varphi}{2\pi} + k\right) m\Psi_{\mathrm{Q}}\right\}_{@k},$$

and an experimental set-up that gave us a constant probability (pseudo-)distribution:

$$\Psi_{\mathrm{M}} = \{\phi_{\mathrm{E}}\Psi_{\mathrm{Q}}\}_{@k} = \{\exp(-2\pi k_1 q)\}.$$

Discussion

The Born rule is satisfied, we made good choices. So far, measurement wave functions

$$\Psi_{\mathrm{M}} = \{\exp(-2\pi k_1 q)\}$$

correspond to eigenvalues

$$\hat{D}\Psi_{\mathrm{Q}} = \left\{\left(\frac{\varphi}{2\pi} + k\right) m\Psi_{\mathrm{Q}}\right\}_{@k}.$$

- How to normalize the Ψ_{M} (or the Ψ_{Q})?
- Should we leave out "unphysical" values of k?
- How about introducing a phase φ_1 to k_1?

Next steps

Address open questions, for example:

- Clarify how to turn the Ψ_{M} into proper distributions (i.e. normalized to 1).
- See how the particle quantum enumerator k relates to the field quantum enumerator k_1.
- Extract actual measurement values.
- Introduce a new "time" dimension, see what the algebra yields.
- Compare these strictly algebraic results with conventional quantum mechanics for clues towards answering the above.

***thanks