A multivalued complex exponentiation
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Intuitive but problematic: Multivalued complex exponentiation from rational exponents

n-th roots of a complex number

The n-th root of a complex number a generally has n possible values. Picking x_1 as one of the roots of a,

$$a, x_1 \in \mathbb{C}, a \neq 0, n \in \mathbb{N}^+,$$
$$x_1^n = a,$$

the other values x_j are:

$$x_j = x_1 \exp\left(i\,\frac{2\pi}{j}\right) \text{ for } j = 2 \dots n.$$

For negative n the roots are the inverse of the positive n-th roots:

$$x_j^n = a \longrightarrow x_j^{-n} = \frac{1}{a} \qquad (n \neq 0)$$

(this works because the complexes are a division algebra).

Rational exponents in general

In general we can define a set X to contain all possible solutions to taking $a \in \mathbb{C}$ to a rational nonzero exponent $m/n \in \mathbb{Q}$:

$$\begin{aligned} X\left(a, \frac{m}{n}\right) &= \left\{a^{\frac{m}{n}}\right\} \\ &= \left\{x_j \mid x_j^n = a^m\right\}. \end{aligned}$$

- Integral powers are repeated multiplication, and are unique.
- This works because the complexes are power-associative,

$$(xx)(xx) = ((xx)x)x \text{ for all } x \in \mathbb{C}.$$

Caution: limit $1^{m/n}$ for $m, n \to \infty$

- In the reals we define $1^0 := 1$ without hesitation.
- However, allowing multivalued $X\left(1, \frac{1}{n}\right) = \left\{1^{\frac{1}{n}}\right\}$, the limit

$$\lim_{n \to \infty} X\left(1, \frac{1}{n}\right)$$

  is a set of numbers that lie infinitely dense on the unit circle.
- For a Cauchy sequence m_j/n_j with $j = 1, 2, 3 \ldots$ that approximates an irrational real number, would $\lim_{j\to\infty}\left\{1^{m_j/n_j}\right\}$ be the *entire* unit circle?
- In this approach, irrational exponents and exponents 0 are problematic.

Another caution: repeated exponentiation

- A product rule for exponentiation,

$$\left(a^b\right)^c \stackrel{??}{=} a^{bc}$$

  with nonzero $a \in \mathbb{C}$, $b, c \in \mathbb{Q}$, generally doesn't exist.
- Example:

$$\left\{\left(1^{\frac{1}{2}}\right)^2\right\} = \{1\}\text{,but } \left\{\left(1^2\right)^{\frac{1}{2}}\right\} = \{+1, -1\}.$$

- Maybe this is not the best approach?

Second try: Complex exponents using exp and log

Define exp from its conventional Taylor polynomial

The exponential function relates geometry and analysis:

$$\begin{aligned} \exp(ix) &= \cos x + i \sin x, \\ \frac{d}{dx}\exp(x) &= \exp(x) \qquad \text{for all } x \in \mathbb{C}. \end{aligned}$$

- Use a definition that suits our requirements so far, to work over a power-associative division algebra with not necessarily commutative or associative multiplication:

$$\exp x := 1 + \sum_{j=1}^{\infty} \frac{1}{j!} \left(\underbrace{x * \ldots * x}_{j \text{ times}} \right).$$

- This expression is single-valued in x.

exp as inverse of log

A multivalued logarithm, log, is defined such that the exponential function is inverse to its solution set:

$$\log x := \{y \mid \exp y = x\}.$$

Using the real logarithm of the absolute[1] of x,

$$\ln|x| \qquad (x \neq 0),$$

phase angles $\varphi_k(x) \in \mathbb{R}$ from arccos principal $\Phi(x)$ with branches

$$\varphi_k(x) = \arccos \frac{\mathrm{Re}(x)}{|\mathrm{Im}(x)|} = \Phi(x) + 2\pi k$$

$(k \in \mathbb{Z})$, and imaginary unit vector i, the logarithm is:

$$\log x = \{\ln|x| + i\varphi_k(x)\} = \{\ln|x| + i(\Phi(x) + 2\pi k)\}.$$

[1] $\mathbb{C}$ is normed, therefore $|x|$ exists.

Example of $z = \exp(\ln z)$ for multivalued z

Let $z = \{z_j\}$, $z_j \in X\left(a, \frac{m}{n}\right)$ multivalued, and identify by definition

$$\exp\{z_j\} \equiv \{\exp z_j\}, \qquad \log\{z_j\} \equiv \{\log z_j\}.$$

- There are no sets of sets, only sets of numbers.
- For example $a = 1$, $m = 1$, $n = 2$:

$$\sqrt{1} = \{1, -1\} = 1^{\frac{1}{2}} = \{\exp(i\pi k)\},$$
$$\log\{1, -1\} = \{\log 1, \log(-1)\} = \left\{\log\left(1^{\frac{1}{2}}\right)\right\} = \left\{\frac{1}{2}\log 1\right\} = \{i\pi k\},$$
$$\exp(\{1, -1\}) = \exp(\{\log 1, \log(-1)\}) = \exp\left(\frac{1}{2}\log 1\right) = \{\exp(i\pi k)\}.$$

Define $z = x^y$ for any $x, y \in \mathbb{C} \setminus \{0\}$

- Drop set notation, all expressions are sets of numbers.
- Generalize for any exponent, and circumvent earlier concerns (e.g. with real exponents in x^y), by defining:

$$\log x^y := (\log x)\, y \qquad (x, y \in \mathbb{C} \setminus \{0\})\,.$$

- Therefore:

$$x^y := \exp((\log x)\, y)\,.$$

The definitions for exp and log require x, y from a power-associative normed division algebra, with not necessarily commutative or associative multiplication. Satisfied by $\mathbb{C}$.

Derivatives of x^y

In $\mathbb{C}$, the derivatives of $x^y := \exp((\log x)\, y)$ then are:

$$\frac{d}{dx}(x^y) = \left(\frac{y}{x}\right) x^y \qquad (x, y \neq 0),$$
$$\frac{d}{dy}(x^y) = \{(\log x)\, x^y\}_{@k}\,.$$

The last line uses notation $\{\ldots\}_{@k}$ to indicate that the same branch k needs to be taken for all factors in the set of products:

$$\{(\log x)\, x^y\}_{@k} :=$$
$$\left\{\left(\ln|x| + i\left(\Phi(x) + 2\pi\, k\right)\right) \exp\left(\left(\ln|x| + i\left(\Phi(x) + 2\pi\, k\right)\right) y\right)\right\}.$$

Product of exponentials

The product of exponentials $x_j^{y_j}$ with $x_j, y_j \in \mathbb{C} \setminus \{0\}$, $j = 1, \ldots, n$ is:

$$\prod_{j=1}^{n} x_j^{y_j} = \exp\left(\sum_{j=1}^{n} (\log x_j)\, y_j\right).$$

- Generally, $x^{y_1} x^{y_2} \neq x^{y_1+y_2}$ due to the different branches of the multivalued logarithm. For example,

$$\begin{aligned} 1^{y_1} 1^{y_2} &= \exp\left((\log 1)\, y_1 + (\log 1)\, y_2\right) \\ &= \exp\left(i\, 2\pi k_1 y_1 + i\, 2\pi k_2 y_2\right) \\ &\neq \exp\left(i\, 2\pi k\, (y_1 + y_2)\right). \end{aligned}$$

- The following hold true:

$$\begin{aligned} \{x^{y_1} x^{y_2}\}_{@k} &= x^{y_1+y_2}, \\ x_1^y x_2^y &= (x_1 x_2)^y. \end{aligned}$$

Select subspaces in x^y for $x, y \in \mathbb{C} \setminus \{0\}$

x^y with $|x|=1$ and $y\in\mathbb{R}$

If x on the unit circle in the complex plane, $x\in S^1$, $|x|=1$,

$$x := \exp(i\alpha) \text{ with } \alpha\in\mathbb{R},$$

then for real $y\in\mathbb{R}$ there is

$$\begin{aligned} x^y &= \exp((\log x)\,y) = \exp((\ln|x| + i(\Phi(x)+2\pi k))\,y) \\ &= \exp(i\,(\alpha+2\pi k)\,y)\,. \end{aligned}$$

- As function of y with x fixed, x^y is an infinite set of numbers that revolve around the origin at varying speeds $\alpha+2\pi k$.
- As function of x with y fixed, the set of numbers $\exp(i\,2\pi k y)$ is rotated around the origin by $\exp(i\,y\alpha)$.

x^{iy} with $|x|=1$ and $y\in\mathbb{R}$

With $x:=\exp(i\alpha)$ on the unit circle and real $y\in\mathbb{R}$ there also is

$$
\begin{aligned}
x^{iy} &= \exp((\log x)\, iy) = \exp((\ln|x| + i(\Phi(x)+2\pi k))\, iy) \\
&= \exp(-(\alpha+2\pi k)\, y) .
\end{aligned}
$$

- The solution set is made of infinitely many real numbers that are exponentially spaced.

x^y with $x \in \mathbb{R} \setminus \{0\}$ for general $y \in \mathbb{C}$

Having a real number in the base yields in general:

$$\begin{aligned} x^y &= \exp\left(\left(\ln|x| + i\left(\Phi(x) + 2\pi k\right)\right) y\right) \\ &= \exp\left(\left(\ln|x| + i\, 2\pi k\right) y\right). \end{aligned}$$

- Real exponents x^y with $x, y \in \mathbb{R} \setminus \{0\}$ are generally multivalued.
- For example, taking the Euler number e to power $y \in \mathbb{C}$ is not the same as the single-valued exponential function:

$$e^y = \exp\left(\left(1 + i\, 2\pi k\right) y\right) \neq \exp(y).$$

- For imaginary y, only the $k = 0$ branch of e^y traces out the unit circle. All other branches spiral inward ($k > 0$) or out ($k < 0$) with increasing $\mathrm{Im}\, y$.

Next steps

The slides showed one possible approach towards multivalued complex exponentiation, in a controlled manner.

Next steps:

- Work out an application for this approach.
- The definition of exponential function and multivalued logarithm are compatible with quaternions $\mathbb{H}$ and octonions $\mathbb{O}$. What are the differentials there? What is the product rule for exponentials in these algebras?

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